Some contraction theorem in Hausdroff space

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ABSTRACT

In this article we prove some fixed point and common fixed point theorems in Hausdroff spaces satisfying new rational type contractive conditions.

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1. INTRODUCTION

The study of common fixed point of mapping contractive type condition has been a very active field of research activity during the last three decades. The most general of the common fixed point pertain to two or three mapping of a metric space \( (X,d) \) and use either a Banach type contractive condition or other contractive condition. Many, Hardy [1], Rajput [2], Yadav [3], Sengupta[4] and so many author work in this field and prove more interesting result. In this paper we establish a theorem to prove the existence of a common fixed point for three mappings [5-15]. Throughout in this paper we denote \( (X,d) \) is a metric space which is denoted simply by \( X \) and \( T:X \rightarrow X \) a selfmap of \( X \).

Let \( (X,d) \) be a metric space. A mapping \( T:X \rightarrow X \) is called a contraction mapping if there exists a positive real constant \( \alpha < 1 \), such that,

\[
d(Tx,Ty) \leq \alpha d(x,y), \quad \forall \ x,y \in X
\]

By the well known Banach contraction principle every contraction mapping of a complete metric space \( (X,d) \) into itself has a unique fixed point. The object of this note is to prove some fixed point theorem in arbitrary topological spaces. Now we prove the following Theorems:

THEOREM 1.1. Let \( T \) be a continuous mapping of a Hausdroff space, \( X \) into itself and let \( d:X \times X \rightarrow R^+ \) be a continuous mapping such that for \( x,y \in X \) and \( x \neq y \) satisfying

\[
\alpha d(Tx,Ty) + \beta (d(x,Tx) + d(y,Ty)) + \gamma (d(x,Ty) + d(y,Tx)) \leq \delta d(x,y)
\]

(1)

Where \( \alpha, \beta, \gamma, \delta \geq 0 \) such that \( \delta < \alpha + 2\beta + 2\gamma \). Then \( T \) has a fixed point in \( X \).
Proof: For any arbitrary \(x_0 \in X\) we choose \(x \in X\), we define a sequence \(\{x_n\}\) of elements of \(X\), such that,

\[
x_{n+1} = Tx_n
\]

For \(n = 0,1,2,3, \ldots\) \n
Now \n
\[
d(x_{n+1},x_{n+2}) = d(Tx_n,Tx_{n+1})
\]

Form (1) \n
\[
a d(Tx_n,Tx_{n+1}) + \beta \{d(x_n,Tx_n) + d(x_{n+1},Tx_{n+1})\} + \gamma \{d(x_n,Tx_{n+1}) + d(x_{n+1},Tx_n)\} \leq \delta d(x_n,x_{n+1})
\]

\[
a d(Tx_n,Tx_{n+1}) + \beta \{d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})\} + \gamma \{d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1})\} \leq \delta d(x_n,x_{n+1})
\]

\[
a d(x_{n+1},x_{n+2}) + \beta \{d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})\} + \gamma \{d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1})\} \leq \delta d(x_n,x_{n+1})
\]

\[
(a + \beta + \gamma) d(x_n,x_{n+1}) \leq (\delta - \beta - \gamma) d(x_n,x_{n+1})
\]

\[
d(x_{n+1},x_{n+2}) \leq \frac{(\delta - \beta - \gamma)}{a + \beta + \gamma} d(x_n,x_{n+1})
\]

Processing the same manner, we get \n
\[
d(x_n,x_{n+1}) \leq \left(\frac{\delta - \beta - \gamma}{a + \beta + \gamma}\right)^n d(x_0,x_1)
\]

\[
\lim_{n \to \infty} d(x_n,x_{n+1}) = 0
\]

\(\{x_n\}\) is converges to its limit \(x\) (say) \n
It is easy to see that, \(\{x_n\}\) is cauchy sequence, which converges to its limit \(x\) (say) \n
We suppose that, \(x \neq Tx\) then, from (1) we have \n
\[
a d(x,Tx) + \beta \{d(x,x) + d(x,Tx)\} + \gamma \{d(x,Tx) + d(x,Tx)\} \leq \delta d(x,x)
\]

Which contradiction. So we have \(x = Tx\). \n
\(x\) is a fixed point of \(T\). \n
Next we prove a common fixed point theorem for two self mapping which is generalization on Theorem 1.1. \n
**Theorem 1.2.** Let \(S,T\) be a continuous mapping of a Hausdroff space, \(X\) into itself and let \(d:X \times X \to \mathbb{R}^+\) be a continuous mapping such that for \(x,y \in X\) and \(x \neq y\). Satisfying condition, \n
\[
a d(Sx,Ty) + \beta \{d(x,Sx) + d(y,Ty)\} + \gamma \{d(x,Ty) + d(y,Sx)\} \leq \delta d(x,y)
\]

(2) \n
Where \(a, \beta, \gamma, \delta \geq 0\) such that \(\delta < a + 2\beta + 2\gamma\). Then \(S,T\) have unique fixed point in \(X\). \n
**Proof:** For and arbitrary \(x_0 \in X\) we choose \(x \in X\), we define a sequence \(\{x_n\}\) of elements of \(X\), such that, \n
\[
x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1}
\]

For \(n = 0,1,2,3, \ldots\) \n
Now, \n
\[
d(x_{2n+1},x_{2n+2}) = d(Sx_{2n},Tx_{2n+1})
\]

From (2), \n
\[
a d(Sx_{2n},Tx_{2n+1}) + \beta \{d(x_{2n},Sx_{2n}) + d(x_{2n+1},Tx_{2n+1})\} + \gamma \{d(x_{2n},Tx_{2n+1}) + d(x_{2n+1},Sx_{2n})\} \leq \delta d(x_{2n},x_{2n+1})
\]

\[
a d(x_{2n+1},x_{2n+2}) + \beta \{d(x_{2n+1},x_{2n+2}) + d(x_{2n+2},x_{2n+1})\} + \gamma \{d(x_{2n+1},x_{2n+2}) + d(x_{2n+2},x_{2n+1})\} \leq \delta d(x_{2n},x_{2n+1})
\]

\[
(a + \beta + \gamma) d(x_{2n+1},x_{2n+2}) \leq (\delta - \beta - \gamma) d(x_{2n},x_{2n+1})
\]

\[
d(x_{2n+1},x_{2n+2}) \leq \frac{(\delta - \beta - \gamma)}{a + \beta + \gamma} d(x_{2n},x_{2n+1})
\]

Proceeding the same manner, we get \n
\[
d(x_{2n+1},x_{2n+2}) \leq \left(\frac{\delta - \beta - \gamma}{a + \beta + \gamma}\right)^{2n+1} d(x_0,x_{2n+1})
\]

On taking \(n \to \infty\) \n
\[
\lim_{n \to \infty} d(x_{2n+1},x_{2n+2}) = 0
\]

That is the sequence \(\{x_n\}\) is convergent.
It is easy to see that, the sequence \( \{x_n\} \) is Cauchy sequence which converges to it limit \( x \) (say).

Now we assume that, \( x \neq Sx \). From (2), \( d(x,Sx) > 0 \)

\[
d(x,x_{2n+2}) + \alpha \{d(x,Sx) + d(x_{2n+1},TX_{2n+1})\} + \beta \{d(x,Tx_{2n+1}) + d(x_{2n+1},Sx)\} \leq \gamma d(x,x_{2n+1})
\]

as \( n \to \infty \) and from the continuity of \( S \)

\[
\lim_{n \to \infty} d(x,Sx) = 0
\]

Which contradiction,

\[
x = Sx
\]

Similarly we show that,

\[
x = Tx
\]

Uniqueness: Let us \( u \) is another fixed point different from \( x \) of \( S \) and \( T \), so that \( u \in X \)

\[
d(x,u) \leq d(x,Sx) + d(Sx,Tx) + d(Tx,u)
\]

By using (2) we can see that,

\[
d(x,u) \leq 0
\]

Which contradiction.

\( x \) is unique common fixed point of \( S \) and \( T \).

**Corollary 1.3:** Let \( S,T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x,y \in X \) and \( x \neq y \). Satisfying condition,

\[
\alpha d(Sx,Ty) \leq \delta d(x,y)
\]

where \( \alpha, \delta \geq 0 \) such that \( \delta < \alpha \). Then \( S,T \) have unique fixed point in \( X \).

**Proof:** If we take \( \beta = \gamma = 0 \) in Theorem 1.2, then we get the result.

**Corollary 1.4:** Let \( S,T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x,y \in X \) and \( x \neq y \). Satisfying condition,

\[
\beta \{d(x,Sx) + d(y,Ty)\} \leq \delta d(x,y)
\]

where \( \beta, \delta \geq 0 \) such that \( \delta < 2 \beta \). Then \( S,T \) have unique fixed point in \( X \).

**Proof:** If we take \( \alpha = \gamma = 0 \), in Theorem 1.2, then we get the result.

**Corollary 1.5:** Let \( S,T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x,y \in X \) and \( x \neq y \). Satisfying condition,

\[
\gamma \{d(x,Ty) + d(y,Sx)\} \leq \delta d(x,y)
\]

where \( \gamma, \delta \geq 0 \) such that \( \delta < 2 \gamma \). Then \( S,T \) have unique fixed point in \( X \).

**Proof:** If we take \( \alpha = \beta = 0 \), in Theorem 1.2, then we get the result.

**Corollary 1.6:** Let \( S,T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x,y \in X \) and \( x \neq y \). Satisfying condition,

\[
\alpha \{d(Sx,Ty) + d(x,Sx) + d(y,Ty) + d(x,Ty) + d(y,Sx)\} \leq \delta d(x,y)
\]

where \( \alpha, \beta, \gamma \delta \geq 0 \) such that \( \delta < 5 \alpha \). Then \( S,T \) have unique fixed point in \( X \).

**Proof:** If we take \( \alpha = \beta = \gamma \), in Theorem 1.2, then we get the result.

Next we prove common fixed point theorem satisfying symmetric rational contractive condition.

**Theorem 1.7:** Let \( S,T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x,y \in X \) and \( x \neq y \). Satisfying condition,

\[
\alpha d(Sx,Ty) + \beta \frac{d(Sx)d(Ty)}{d(Sx)} + \gamma \frac{d(x,y)}{d(Sx)} \leq \delta d(x,y)
\]

where \( \alpha, \beta, \gamma \delta \geq 0 \) such that \( \delta < \alpha + \beta + \gamma \). Then \( S,T \) have unique fixed point in \( X \).

**Proof:** For and arbitrary \( x_0 \in X \) we choose \( x \in X \), we define a sequence \( \{x_n\} \) of elements of \( X \), such that,
\[ x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \]

For \( n = 0, 1, 2, 3, \ldots \).

Now,
\[ d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{n+1}) \]

From (7),
\[ \alpha \ d(x_{2n}, Tx_{n+1}) + \beta \ \frac{d(x_{2n}, Sx_{2n})}{d(x_{2n}, Tx_{n+1})} + \gamma \ \frac{d(x_{2n}, Sx_{2n})}{d(x_{2n}, Tx_{n+1})} \leq \delta d(x_{2n}, x_{2n+1}) \]
\[ \alpha d(x_{2n+1}, x_{2n+2}) + \beta \ \frac{d(x_{2n+1}, x_{2n+2})}{d(x_{2n+1}, Sx_{2n+2})} + \gamma \ \frac{d(x_{2n+1}, x_{2n+2})}{d(x_{2n+1}, Sx_{2n+2})} \leq \delta d(x_{2n}, x_{2n+1}) \]

Proceeding the same manner, we get
\[ d(x_{2n+1}, x_{2n+2}) \leq \left( \frac{\delta - \beta}{\alpha} \right) d(x_{2n}, x_{2n+1}) \]

On taking \( n \to \infty \),
\[ \lim_{n \to \infty} d(x_{2n+1}, x_{2n+2}) \to 0 \]

That is the sequence \( \{x_n\} \) is convergent.

It is easy to see that, the sequence \( \{x_n\} \) is Cauchy sequence which converges to limit \( x \) (say).

Now we assume that, \( x \neq Sx \). From (7), \( d(x, Sx) > 0 \)
\[ \alpha d(x, x_{2n+2}) + \beta \ \frac{d(x, Sx)}{d(x, x_{2n+2})} + \gamma \ \frac{d(x, Sx)}{d(x, x_{2n+2})} \leq \delta d(x, x_{2n+1}) \]

as \( n \to \infty \) and from the continuity of \( S \),
\[ \lim_{n \to \infty} d(x, Sx) \to 0 \]

Which contradiction,
\[ x = Sx \]

Similarly we show that,
\[ x = Tx \]

Uniqueness; Let us \( u \) is another fixed point different from \( x \) of \( S \) and \( T \), so that \( u \in X \)
\[ d(x, u) \leq d(x, Sx) + d(Sx, Tu) + d(Tu, u) \]

By using (7) we can see that,
\[ d(x, u) \leq 0 \]

Which contradiction.

\( x \) is unique common fixed point of \( S \) and \( T \).

**Corollary 1.8**: Let \( S, T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x, y \in X \) and \( x \neq y \). Satisfying condition,
\[ \alpha d(Sx, Ty) + \beta \ \frac{d(x, Sx)}{d(Sx, Ty)} \leq \delta d(x, y) \]  \hspace{1cm} (8)

Where \( \alpha, \beta, \delta > 0 \) such that \( \delta < \alpha + \beta \). Then \( S, T \) have unique fixed point in \( X \).

**Proof**: If we take \( \gamma = 0 \), in Theorem 1.7, then we get the result.

**Corollary 1.9**: Let \( S, T \) be a continuous mapping of a Hausdorff space, \( X \) into itself and let \( d: X \times X \to R^+ \) be a continuous mapping such that for \( x, y \in X \) and \( x \neq y \). Satisfying condition,
\[ \alpha d(Sx, Ty) + \gamma \ \frac{d(x, Ty)}{d(Sx, Ty)} \leq \delta d(x, y) \]  \hspace{1cm} (9)

Where \( \alpha, \gamma, \delta > 0 \) such that \( \delta < \alpha \). Then \( S, T \) have unique fixed point in \( X \).
Proof: If we take $\beta = 0$, in Theorem 1.7, then we get the result.

Corollary 1.10: Let $S, T$ be a continuous mapping of a Hausdorff space, $X$ into itself and let $d: X \times X \to R^+$ be a continuous mapping such that for $x, y \in X$ and $x \neq y$. Satisfying condition,
\[
\alpha \left( \frac{d(Sx, Ty)}{d(x, y)} + \frac{d(x, Ty)}{d(x, y)} + \frac{d(y, Sy)}{d(x, y)} \right) \leq \delta (x, y)
\]  
Where $\alpha, \delta > 0$ such that $\delta < 2\alpha$. Then $S, T$ have unique fixed point in $X$.

Proof: If we take $\alpha = \beta = \gamma$, in Theorem 1.7, then we get the result.

Theorem 1.11. Let $S, T$ be a continuous mapping of a Hausdorff space, $X$ into itself and let $d: X \times X \to R^+$ be a continuous mapping such that for $x, y \in X$ and $x \neq y$. Satisfying condition,
\[
\alpha d(Sx, Ty) + \beta d^2(x, Sx) + \gamma d^2(y, Ty) \leq \delta d(x, y)
\]  
Where $\alpha, \beta, \gamma, \delta \geq 0$ such that $\delta < \alpha + 2\beta + 2\gamma$. Then $S, T$ have unique fixed point in $X$.

Proof: For and arbitrary $x_0 \in X$ we choose $x \in X$, we define a sequence $\{x_n\}$ of elements of $X$, such that,
\[
x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1}
\]
For $n = 0, 1, 2, 3, \ldots$

Now,
\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})
\]
From (11),
\[
\alpha d(Sx_{2n}, Tx_{2n+1}) + \beta \frac{d^2(x_{2n}, Sx_{2n}) + d^2(x_{2n}, Tx_{2n+1})}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1})} \leq \delta d(x_{2n}, x_{2n+1})
\]
\[
\alpha d(x_{2n+1}, x_{2n+2}) + \beta [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \leq \delta d(x_{2n}, x_{2n+1})
\]
\[
(\alpha + \beta + \gamma) d(x_{2n+1}, x_{2n+2}) \leq (\delta - \beta - \gamma) d(x_{2n}, x_{2n+1})
\]
\[
d(x_{2n+1}, x_{2n+2}) \leq \frac{\delta - \beta - \gamma}{\alpha + \beta + \gamma} d(x_{2n}, x_{2n+1})
\]

Proceeding the same manner, we get
\[
d(x_{2n+1}, x_{2n+2}) \leq \left[ \frac{\delta - \beta - \gamma}{\alpha + \beta + \gamma} \right]^{n+1} d(x_{2n}, x_{2n+1})
\]

On taking $n \to \infty$
\[
\lim_{n \to \infty} d(x_{2n+1}, x_{2n+2}) = 0
\]

That is the sequence $\{x_n\}$ is convergent.

It is easy to see that, the sequence $\{x_n\}$ is Cauchy sequence which converges to it limit $x$ (say).

Now we assume that, $x \neq Sx$. From (11), $d(x, Sx) > 0$
\[
\alpha d(x, x_{2n+1}) + \beta [d(x, Sx) + d(x_{2n+1}, Tx_{2n+1})] + \gamma [d(x, Sx) + d(x_{2n+1}, Sx)] \leq \delta d(x, x_{2n+1})
\]

as $n \to \infty$ and from the continuity of $S$
\[
\lim_{n \to \infty} d(x, Sx) = 0
\]
Which contradiction,
\[
x = Sx
\]

Similarly we show that,
\[
x = Tx
\]

Uniqueness; Let us $u$ is another fixed point different from $x$ of $S$ and $T$, so that $u \in X$
\[
d(x, u) \leq d(x, Sx) + d(Sx, Tu) + d(Tu, u)
\]

By using (2) we can see that,
\[
d(x, u) \leq 0
\]
Which contradiction.

$x$ is unique common fixed point of $S$ and $T$.
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