Some new contraction for Coupled Fixed Point Theorems on G- metric space

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ABSTRACT

In this paper, we find a new type of contractive condition on G- metric spaces also the purpose of this paper is to generalized some recent coupled fixed point theorems in G- metric spaces. We further give some concrete examples in order to support our main theorems. The obtained results generalize those announced by many authors.

Subject classification [2000]: 54H25, 46J10

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1. INTRODUCTION

Mustafa and Sims introduced a new notion of generalized metric space denoted G-metric space [11], a generalization of the metric space $(X,d)$, to develop and a new fixed point theory for a variety of mappings and to extend known metric space theorems to a more general setting. Subsequently several fixed point results were proven in these spaces [1,2,12,13,14,15].

We present now the necessary definitions and results in G- metric spaces, which will be useful for the rest.

Definition- 1 [11] Let $X$ be a non- empty set and $G: X \times X \times X \rightarrow R^+$ be a function satisfying the following properties:

i. $G(x,y,z) = 0$ if $x = y = z$

ii. $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$.

iii. $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $x \neq y$.

iv. $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables).

v. $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric, or, more specially, a $G$ – metric on $X$, and the pair $(X,G)$ is called a G- metric space.

Definition- 2 [11] Let $(X,G)$ be a G- metric space, and let $\{x_n\}$ be a sequence of points of $X$, therefore, we say that $\{x_n\}$ is G- convergent to $x \in X$ if $\lim_{n \to \infty} = G(x,x_n,x_m) = 0$, that is, for any $\epsilon > 0$ , there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geq N$. We call $x$ the limit of the sequence and write $x_n \rightarrow x$ or $\lim_{n \to \infty} x_n = x$.

Lemma-3 [11] Let $(X,G)$ be a G- metric space. The following statements are equivalent:

i. $\{x_n\}$ is G- convergent to $x$, 

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\[ \begin{align*}
\text{Definition-4[11]} & \quad \text{Let } (X, G) \text{ be a G- metric space. A sequence } \{x_n\} \text{ is called a G- Cauchy sequence if, for any } \varepsilon > 0, \text{ there exists } n \in N \text{ such that } G(x_n, x_m) < \varepsilon \text{ for all } n, m, l \geq N, \text{ that is, } G(x_n, x_m) \to 0 \text{ as } n, m, l \to +\infty, \\
\text{Lemma-5[11]} & \quad \text{Let } (X, G) \text{ be a G- metric space. The following statements are equivalent:} \\
\text{i. } & \quad \text{The sequence } \{x_n\} \text{ is G- Cauchy,} \\
\text{ii. } & \quad \text{for any } \varepsilon > 0, \text{ there exists } n \in N \text{ such that } G(x_n, x_m) < \varepsilon \text{ for all } n, m, l \geq N. \\
\text{Definition-6[11]} & \quad \text{A G- metric space } (X, G) \text{ is called G- complete if every G- Cauchy sequence is G- convergent in } (X, G). \\
\text{Every G- metric on } X \text{ defines a metric } d_G \text{ on } X \text{ given by} \\
d_G(x, y) = 2G(x, x) + G(y, x, x) \text{ for all } x, y \in X. \\
\text{Lemma-7[11]} & \quad \text{If } X \text{ is a G- metric space, then } G(x, y, z) = 2G(y, x, x) \text{ for all } x, y, z \in X. \\
\text{Lemma-8[11]} & \quad \text{If } X \text{ is a G- metric space, then } G(x, x, y) = G(x, x, z) + G(x, x, y) \text{ for all } x, y, z \in X. \\
\text{In recent time, fixed point theory has been developed rapidly in partially ordered metric space. Bhaskar and} \\
\text{Lakshmikantham [5] introduced the concept of mixed monotone property. Furthermore, they proved some coupled} \\
\text{fixed point theorems for mapping which satisfy the mixed monotone property, and gave a beautiful application in} \\
\text{the existence of a solution for a periodic boundary value problem. This concept follows,} \\
\text{Definition-9} & \quad \text{Let } (X, \leq) \text{ is a partially ordered set and } F : X \times X \to X. \text{ The mapping } F \text{ is said to have the mixed} \\
\text{monotone property if } F \text{ is nondecreasing monotone in its first argument and is a nonincreasing monotone in its} \\
\text{second argument, that is, for any } x, y \in X \\
x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \quad (1.1) \\
\text{and} \\
y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2) \quad (1.2) \\
\text{Lakshmikantham and Cirić [9] generalized the concept of mixed monotone mapping and proved a common coupled} \\
\text{fixed point theorem using the following concept of mixed g- monotone mapping,} \\
\text{Definition-10} & \quad \text{Let } (X, \leq) \text{ is a partially ordered set and } F : X \times X \to X \text{ and } g : X \to X. \text{ The mapping } F \text{ is said to have} \\
\text{the mixed g- monotone property if } F \text{ is g- nondecreasing monotone in its first argument and is g- nonincreasing} \\
\text{monotone in its second argument, that is, for any } x, y \in X \\
x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y) \quad (1.3) \\
\text{and} \\
y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2) \quad (1.4) \\
\text{Definition – [10] reduces to Definition – [9] when } g \text{ is the identity mapping.} \\
\text{Definition-11} & \quad \text{Let } X \text{ be a non empty set and } F : X \times X \to X \text{ is said to be continuous if for any two G- convergent} \\
\text{sequences } \{x_n\} \text{ and } \{y_n\} \text{ which converges to } x \text{ and } y \text{ respectively, } \{F(x_n, y_n)\} \text{ is G- convergent to } F(x, y). \\
\text{Definition-12} & \quad \text{Let } X \text{ be a non empty set and } F : X \times X \to X \text{ and } g : X \to X \text{ two mappings. } F \text{ and } g \text{ are commutative if} \\
g(F(x, y)) = F(g(x), g(y)), \forall x, y \in X. \\
\text{By using the concept of mixed monotone and mixed g- monotone mapping we prove a coupled fixed point theorem} \\
\text{which is generalization of many existing coupled fixed point results on G- metric spaces [11]. We also give an} \\
\text{example in support of our result.} \\
\text{Now in next section we give some previous known results on G- metric space.}
2. PRELIMINARIES

Denote $\Phi$ be the set of functions $\phi: [0, \infty) \to [0, \infty)$ satisfying the following conditions,

i. $\phi$ is continuous and non-decreasing,

ii. $\phi(t) = 0$ if and only if $t = 0$,

iii. $\phi(\alpha t) \leq \alpha \phi(t)$ for $\alpha \in [0, \infty)$, and

iv. $\phi(t + s) \leq \phi(t) + \phi(s)$ for all $s, t \in [0, \infty)$.

Also, let $\Psi$ be the set of all functions $\psi: [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the condition $\lim_{t_1 \to r_1, t_2 \to r_2} \psi(t_1, t_2) \geq 0$ for all $(r_1, r_2) \in [0, \infty) \times [0, \infty)$ with $r_1 + r_2 > 0$. For example

i. $\psi(t_1, t_2) = k \max \{t_1, t_2\}$ for some $k \in [0, 1)$,

ii. $\psi(t_1, t_2) = \alpha t_1^p + \beta t_2^q$ for $\alpha, \beta, p, q > 0$, and

iii. $\psi(t_1, t_2) = \frac{1}{k}(t_1 + t_2)$ for some $k \in [0, 1)$.

Choudhury and Malty [6] gave the first result of coupled fixed point theory. They studied necessary and sufficient conditions for the existence of coupled fixed point in partial ordered $G$-metric spaces and obtained the following interesting result on $G$-metric space.

**Theorem-13 [6]** Let $(X, \leq)$ be a partially ordered set such that $X$ is a complete $G$-metric space and $F: X \times X \to X$ be a mapping having the mixed monotone property on $X$. Suppose that there exists $k \in [0, 1)$ such that

$$
G(F(x, y), F(u, v), F(w, z)) \leq \frac{1}{k} \left( G(x, u, w) + G(y, v, z) \right)
$$

for all $x, y, u, v, w, z \in X$ where $x \geq u \geq w$ and $y \leq v \leq z$, where either $u \neq w$ or $v \neq z$. If there exists $x_0, y_0 \in X$ such that

$$
x_0 \leq F(x_0, y_0), \quad y_0 \leq F(y_0, x_0)
$$

and either

i. $F$ is continuous or

ii. $X$ has the following property:

a. if a non-decreasing sequence $\{x_n\}$ such that $x_n \to x$ then $x_n \leq x$ for all $n$,

b. if a non-increasing sequence $\{y_n\}$ such that $y_n \to y$ then $y_n \geq y$ for all $n$, then $F$ has a coupled fixed point.

Aydi et al. [3] generalized this by using the altering distance function and proved the following coupled common fixed point theorem on $G$-metric space.

**Theorem-14 [3]** Let $(X, \leq)$ be a partially ordered set such that $X$ is a complete $G$-metric space. Suppose that there exist $\phi \in \Phi$ and $F: X \times X \to X$ and $g: X \to X$ such that

$$
G(F(x, y), F(u, v), F(w, z)) \leq \phi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right)
$$

for all $x, y, u, v, w, z \in X$ where $x \geq u \geq w$ and $y \leq v \leq z$. Suppose also that $F$ is continuous and has the mixed monotone property. $F(X \times X) \subseteq g(X)$ and $g$ is continuous and commutes with $F$. If there exists $x_0, y_0 \in X$ such that

$$
gx_0 \leq F(x_0, y_0) \quad \text{and} \quad gy_0 \geq F(y_0, x_0)
$$

then $F$ and $g$ have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Beside this by using basically concept, Luong and Thuan [10] presented the following coupled fixed point theorem for nonlinear contractive type mappings having the mixed monotone property in partial ordered $G$-metric spaces.

**Theorem-15 [10]** Let $(X, \leq)$ be a partially ordered set such that $X$ is a complete $G$-metric space and $F: X \times X \to X$ be a mapping having the mixed monotone property on $X$. Suppose that there exists $\psi \in \Psi$ such that

$$
G(F(x, y), F(u, v), F(w, z)) \leq \frac{G(x, u, w) + G(y, v, z)}{2} - \psi(G(x, u, w), G(y, v, z))
$$

for all $x, y, u, v, w, z \in X$ where $x \geq u \geq w$ and $y \leq v \leq z$, where either $u \neq w$ or $v \neq z$. If there exists $x_0, y_0 \in X$ such that

$$
x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0)
$$


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and either 
   i. F is continuous or 
   ii. X has the following property: 
       a. if a non decreasing sequence \( \{ x_n \} \) such that \( x_n \rightarrow x \) then \( x_n \leq x \) for all \( n \), 
       b. if a non increasing sequence \( \{ y_n \} \) such that \( y_n \rightarrow y \) then \( y_n \geq y \) for all \( n \), 

then F has a coupled fixed point.

Nashine [16] introduced a new contractive condition for coupled fixed point theorem in G- metric space and proved the following coupled fixed point result on G- metric spaces.

**Theorem-16** [16] Let \((X, G, \preceq)\) be a partially ordered G-metric space. Let \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) be mappings such that F has the mixed \( g \)-monotone property, and let there exist \( x_0, y_0 \in X \) such that \( gx_0 \leq F(x_0, y_0) \) and \( F(y_0, x_0) \leq gy_0 \). Suppose that there exists \( k \in [0, 1) \) such that for all \( x, y, u, v, w, z \in X \),

\[
G(F(x, y), F(u, v), F(w, z)) + G(F(y, x), F(v, u), F(z, w)) \leq k [G(gx, gu, gw) + G(gy, gv, gz)] 
\] (2.4)

Assume the following hypotheses:
   i. \( F(X \times X) \subseteq g(X) \), 
   ii. \( g(X) \) is \( G \)- complete, 
   iii. \( g \) is \( G \)-continuous and commutes with \( F \).

then F and g have a coupled coincidence point, that is, there exists \( (x, y) \in X \times X \) such that \( gx = F(x, y) \) and \( gy = F(y, x) \).

Karapnar et al. [7] generalized this result and proved the following common coupled fixed point theorem in G- metric spaces.

**Theorem-17** [7] Let \((X, G, \preceq)\) be a partially ordered G-metric space. Let \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) be mappings such that F has the mixed \( g \)-monotone property, and let there exist \( x_0, y_0 \in X \) such that \( gx_0 \leq F(x_0, y_0) \) and \( F(y_0, x_0) \leq gy_0 \). Suppose that there exists \( k \in [0, 1) \) such that for all \( x, y, u, v, w, z \in X \),

\[
G(F(x, y), F(u, v), F(w, z)) + G(F(y, x), F(v, u), F(z, w)) \leq k [G(gx, gu, gw) + G(gy, gv, gz)] 
\] (2.5)

while 
   i. \( F(X \times X) \subseteq g(X) \), 
   ii. \( g(X) \) is \( G \)- complete, and 
   iii. \( g \) is \( G \)-continuous and commutes with \( F \).

then F and g have a coupled coincidence point, that is, there exists \( (x, y) \in X \times X \) such that \( gx = F(x, y) \) and \( gy = F(y, x) \).

Wangkeeree and Bantaogai [18] proved the following common coupled fixed point theorems which is generalization of Theorem-13.

**Theorem-18** [18] Let \((X, \preceq)\) be a partially ordered set such that \( X \) is a complete G- metric space and \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) be mappings having the mixed \( g \)-monotone property on \( X \). Suppose that there exists \( \psi \in \Psi \) such that 

\[
G(F(x, y), F(u, v), F(w, z)) \leq G(gx, gu, gw) + G(gy, gv, gz) - 2 \psi(G(gx, gu, gw), G(gy, gv, gz)) 
\] (2.6)

for all \( x, y, u, v, w, z \in X \) with \( gx \geq gu \geq gw \) and \( gy \leq gv \leq gz \). If there exists \( x_0, y_0 \in X \) such that \( gx_0 \leq F(x_0, y_0) \) and \( gy_0 \geq F(y_0, x_0) \)

where \( F : X \times X \subseteq g(X) \), \( g \) is continuous and commutes with \( F \), and either
   i. \( F \) is continuous or 
   ii. \( X \) has the following property: 
       a. if a non decreasing sequence \( \{ x_n \} \) such that \( x_n \rightarrow x \) then \( x_n \leq x \) for all \( n \), 
       b. if a non increasing sequence \( \{ y_n \} \) such that \( y_n \rightarrow y \) then \( y_n \geq y \) for all \( n \), 

then F and g have a coupled coincidence point, that is, there exists \( (x, y) \in X \times X \) such that 

\[
g(x) = F(x, y) \text{ and } g(y) = F(y, x). 
\]
3. COUPLED COINCIDENCE POINTS

The main result in this paper is the following coincidence point theorem which generalizes Theorems [13,14,15,16,17,18]

**Theorem-19** Let \((X, \leq)\) be a partially ordered set such that \(X\) is a complete \(g\)-metric space and \(F: X \times X \rightarrow X\) and \(g: X \rightarrow X\) be mappings having the mixed \(g\)-monotone property on \(X\). Suppose that there exists \(\psi \in \Psi\) and \(\phi \in \Phi\) such that

\[
M(x,y,z,u,v,w) \leq \phi\left(\frac{g(x,gx,gw)+g(y,gy,gv)}{2}\right) - 2\psi(G(g(x,u,w),G(gy,v,gz)) (3.1)
\]

where

\[
M(x,y,z,u,v,w) = aG(F(x,y),F(u,v),F(w,z)) + bG(F(y,x),F(v,u),F(z,w))
\]

for all \(a,b \in (0,\infty)\) and \(x,y,z,u,v,w \in X\) for which \(gx \geq gu \geq gw\) and \(gy \leq gv \leq gz\). If there exists \(x_0,y_0 \in X\) such that

\[
gx_0 \leq F(x_0,y_0)\]

and \(gy_0 \geq F(y_0,x_0)\)

and where \(F: (X \times X) \subseteq g(X)\), \(g\) is continuous and commutes with \(F\), and either

i. \(F\) is continuous or

ii. \(X\) has the following property:

a. if a non decreasing sequence \(\{x_n\}\) such that \(x_n \rightarrow x\) then \(x_n \leq x\) for all \(n\),

b. if a non increasing sequence \(\{y_n\}\) such that \(y_n \rightarrow y\) then \(y_n \geq y\) for all \(n\),

then \(F\) and \(g\) have a coupled coincidence point, that is, there exists \((x,y) \in X \times X\) such that

\[
g(x) = F(x,y)\]

and \(g(y) = F(y,x)\).

**Proof:** Let \(x_0,y_0 \in X\) satisfy \(gx_0 \leq F(x_0,y_0)\) and \(gy_0 \geq F(y_0,x_0)\). Since \(F: (X \times X) \subseteq g(X)\), we can choose \(gx_1,gy_1 \in X\) such that \(gx_1 = F(x_0,y_0)\) and \(gy_0 = F(y_0,x_0)\). Again since \(F: (X \times X) \subseteq g(X)\), we can choose \(x_2,y_2 \in X\) such that \(gx_2 = F(x_1,y_1)\) and \(gy_2 = F(y_1,x_1)\). Continuing this process, we can construct sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
g(x_{n+1}) = F(x_n,y_n)\]

and \(g(y_{n+1}) = F(y_n,x_n) \forall n \geq 0\). (3.2)

Next, we show that

\[
g(x_n) \leq g(x_{n+1})\]

and \(g(y_n) \geq g(y_{n+1}) \forall n \geq 0\). (3.3)

Since \(g(x_n) \leq F(x_n,y_n) = g(x_{n+1})\) and \(g(y_n) \leq F(y_n,x_n) = g(y_{n+1})\), therefore, (3.3) holds for \(n = 0\). Next, suppose that (3.3) holds for some fixed \(n \geq 0\), that is,

\[
g(x_n) \leq g(x_{n+1})\]

and \(g(y_n) \geq g(y_{n+1})\) (3.4)

Since \(F\) is the mixed \(g\)-monotone property, from (3.4) and (1.3), imply that

\[
F(x_{n-1},y) \leq F(x_{n-1},y)\]

and \(F(y,x_{n-1}) \leq F(y,x_{n-1})\) (3.5)

for all \(x,y \in X\). Consequently (3.4) and (1.3) refer that

\[
F(y,x_{n-1}) \geq F(y,x_{n-1})\]

and \(F(x,y_{n-1}) \geq F(x,y_{n-1})\) (3.6)

for all \(x,y \in X\). If we substitute \(y = y_n\) and \(x = x_n\) in (3.5), then we obtain

\[
g(x_{n+1}) = F(x_n,y_n) \leq F(x_{n+1},y_{n+1})\]

and \(F(y_{n+1},x_{n+1}) \leq F(y_{n+1},x_{n+1})\) (3.7)

If we take \(y = y_{n+1}\) and \(x = x_{n+1}\) in (3.6) then

\[
F(y_{n+1},x_{n+1}) \geq F(y_{n+1},x_{n+1}) = g(x_{n+2})\]

and \(g(x_{n+2}) = F(x_{n+1},y_{n+1})\) (3.8)

Now, from (3.7) and (3.8), we have

\[
g(x_{n+1}) \leq g(x_{n+2})\]

and \(g(y_{n+1}) \geq g(y_{n+2})\) (3.9)

By the mathematical induction, we conclude that (3.3) holds for all \(n \geq 0\). Since \(g(x_n) \leq g(x_{n+1})\) and \(g(y_n) \geq g(y_{n+1})\) for all \(n \geq 0\), (3.1) implies that

\[
M(x_n,x_{n-1},y_n,y_{n-1}) \leq \phi\left(\frac{g(x_n,gx_{n+1},gx_{n+1})+g(y_n,gy_{n+1},gy_{n+1})}{2}\right)

- 2\psi(G(g(x_n,x_{n+1}),G(gy_n,gy_{n+1},gy_{n+1})) (3.10)

where

\[
M(x_n,x_{n-1},y_n,y_{n-1}) = aG(F(x_n,y_n),F(x_{n-1},y_{n-1})) + bG(F(y_n,x_n),F(y_{n-1},x_{n-1})).
\]
Setting

\[ w_{n+1}^x = G(gx_{n+1}, gx_{n+1}, gx_n) \forall n \geq 0 \]

and

\[ w_{n+1}^y = G(gy_{n+1}, gy_{n+1}, gy_n) \forall n \geq 0 \]

in (3.10), we obtain

\[ a w_{n+1}^x + b w_{n+1}^y \leq \phi \left( \frac{w_n^x + w_n^y}{2} \right) - 2 \psi (w_n^x, w_n^y). \tag{3.11} \]

As \( \psi(t_1, t_2) \geq 0 \) for all \( (t_1, t_2) \in [0, \infty) \times [0, \infty) \) we have

\[ a w_{n+1}^x + b w_{n+1}^y \leq a w_n^x + b w_n^y, \forall n \geq 0 \]

Then the sequence \( \{ w_n^x + w_n^y \} \) is decreasing. Therefore, there exists \( w \geq 0 \) such that

\[ \lim_{n \to \infty} (a w_n^x + b w_n^y) = \lim_{n \to \infty} (aG(gx_{n+1}, gx_{n+1}, gx_n) + bG(gy_{n+1}, gy_{n+1}, gy_n)) \]

or

\[ \lim_{n \to \infty} (aw_n^x + bw_n^y) = (a+b)w \tag{3.12} \]

Now, we show by contradiction that \( w = 0 \). Suppose that \( w > 0 \). From (3.12) the sequences \( \{ G(gx_{n+1}, gx_{n+1}, gx_n) \} \) and \( \{ G(gy_{n+1}, gy_{n+1}, gy_n) \} \) have convergent subsequences \( \{ G(gx_{n_j+1}, gx_{n_j+1}, gx_{n_j}) \} \) and \( \{ G(gy_{n_j+1}, gy_{n_j+1}, gy_{n_j}) \} \), respectively. Assume that

\[ \lim_{j \to \infty} a w_{n_j}^x = \lim_{j \to \infty} (G(gx_{n+1}, gx_{n+2}, gx_n)) = a w_1 \]

and

\[ \lim_{j \to \infty} b w_{n_j}^y = \lim_{j \to \infty} (G(gy_{n+1}, gy_{n+2}, gy_n)) = b w_2 \]

which gives that \( aw_1 + bw_2 = (a+b)w \). From (3.11), we have

\[ a w_{n+1}^n + b w_{n+1}^y \leq \phi \left( \frac{w_n^x + w_n^y}{2} \right) - 2 \psi (w_n^x, w_n^y). \tag{3.13} \]

Then taking the limit as \( j \to \infty \) in the above inequality, we obtain

\[ (a+b)w \leq \phi \left( \frac{2}{2} \right) - 2 \lim_{j \to \infty} \psi (w_{n_j}^x, w_{n_j}^y) < (a+b)w \tag{3.14} \]

which is contradiction. Thus \( w = 0 \), that is

\[ \lim_{n \to \infty} (aw_n^x + bw_n^y) = 0 \tag{3.15} \]

Next, we show that \( \{ g(x_n) \} \) and \( \{ g(y_n) \} \) are G- Cauchy sequences. On the contrary, assume that at least one of \( \{ g(x_n) \} \) or \( \{ g(y_n) \} \) is not a G- Cauchy sequence. By Lemma -5 there is an \( \epsilon > 0 \) for which we can find subsequences \( \{ g(x_{n_k}) \}, \{ g(x_{m_k}) \} \) of \( \{ g(x_n) \} \) and \( \{ g(y_{n_k}) \}, \{ g(y_{m_k}) \} \) of \( \{ g(y_n) \} \) with \( n(k) > m(k) \geq k \) such that

\[ G\left( g(x_{n_k}), g(x_{m_k}) \right) + G\left( g(y_{n_k}), g(y_{m_k}) \right) \geq \epsilon \tag{3.16} \]

Further corresponding to \( m(k) \) we can choose \( n(k) \) in such a way that it is the smallest integer with \( n(k) > m(k) \geq k \) and satisfies (3.16) Then

\[ G\left( g(x_{n(k)-1}), g(x_{n(k)}) \right) \leq G\left( g(x_{n(k)-1}), g(x_{n(k)-1}) \right) \tag{3.17} \]

By Lemma -5, we have

\[ G\left( g(x_{n(k)}), g(x_{m(k)}) \right) \leq G\left( g(x_{n(k)}), g(x_{n(k)-1}) \right) \]

\[ + G\left( g(x_{m(k)-1}), g(x_{m(k)-1}) \right) \tag{3.18} \]

and

\[ G\left( g(y_{n(k)}), g(y_{m(k)}) \right) \leq G\left( g(y_{n(k)}), g(y_{n(k)-1}) \right) \]

\[ + G\left( g(y_{m(k)-1}), g(y_{m(k)-1}) \right) \tag{3.19} \]

from (3.16)–(3.19) we have

\[ \epsilon \leq G\left( g(x_{n(k)}), g(x_{n(k)}) \right) + G\left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) \]

\[ \leq G\left( g(x_{n(k)}), g(x_{n(k)}) \right) + G\left( g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)-1}) \right) \]

\[ + G\left( g(x_{n(k)-1}), g(x_{n(k)-1}) \right) + G\left( g(y_{n(k)-1}), g(y_{n(k)-1}) \right) \]

\[ < G\left( g(x_{n(k)}), g(x_{n(k)}) \right) + G\left( g(y_{n(k)}), g(y_{n(k)}) \right) + \epsilon. \]
Then letting \( k \to \infty \) in the above inequality and using (3.15), we have
\[
\lim_{k \to \infty} \left[ G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \right] = \epsilon. \tag{3.20}
\]
By Lemma-7 and Lemma-8, we have
\[
\begin{align*}
\begin{split}
a &\cdot G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) \\
&\leq a \cdot G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)+1})) \\
&\quad + a \cdot G(g(x_{n(k)+1}), g(x_{n(k)+2}), g(x_{m(k)})) \\
&\quad \leq 2a \cdot G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{n(k)})) \\
&\quad + a \cdot G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)})) \\
&\quad + a \cdot G(g(x_{m(k)+1}), g(x_{m(k)+1}), g(x_{m(k)})) \tag{3.21}
\end{split}
\end{align*}
\]
and
\[
\begin{align*}
\begin{split}
b &\cdot G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \\
&\leq b \cdot G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)+1})) \\
&\quad + b \cdot G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)})) \\
&\quad \leq 2b \cdot G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{n(k)})) \\
&\quad + b \cdot G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)})) \\
&\quad + b \cdot G(g(y_{m(k)+1}), g(y_{m(k)+1}), g(y_{m(k)})) \tag{3.22}
\end{split}
\end{align*}
\]
It follows from (3.21) and (3.22) that
\[
\begin{align*}
\begin{split}
a &\cdot G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + b \cdot G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \\
&\leq 2\left( (a \cdot w_{n[k]+1}^x + b \cdot w_{n[k]+1}^y) + \left( (a \cdot w_{m[k]+1}^x + b \cdot w_{m[k]+1}^y) ight) \right) \\
&+ G(g(x_{n(k)+1}), g(x_{n(k)+2}), g(x_{m(k)})) \\
&\leq \phi \left( G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \right) \tag{3.23}
\end{split}
\end{align*}
\]
Since \( n(k) \geq m(k) \), we get
\[
G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) \text{ and } G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \leq G(g(x_{m(k)}), g(x_{m(k)}), g(y_{m(k)}))
\]
also from (3.1)
\[
\begin{align*}
\begin{split}
a &\cdot G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)})) + b \cdot G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)})) \\
&= a \cdot G(F(x_{n(k)}), F(y_{n(k)}), F(x_{m(k)})) \\
&\quad + b \cdot G(F(y_{n(k)}), F(x_{n(k)}), F(y_{m(k)})) \\
&\leq \phi \left( G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \right) \tag{3.24}
\end{split}
\end{align*}
\]
In view of (3.23) and (3.24), we have
\[
\begin{align*}
\begin{split}
2\left( (a \cdot w_{n[k]+1}^x + b \cdot w_{n[k]+1}^y) + \left( (a \cdot w_{m[k]+1}^x + b \cdot w_{m[k]+1}^y) \right) \right) \\
&\geq a \cdot G(g(x_{n(k)}, g(x_{n(k)}), g(x_{m(k)})) \\
&\quad + b \cdot G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \\
&\quad - a \cdot G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)})) \\
&\quad - b \cdot G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)})) \\
&\quad \geq a \cdot G(g(x_{n(k)}, g(x_{n(k)}), g(x_{m(k)})) \\
&\quad + b \cdot G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \\
&\quad - \phi \left( G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \right) \\
&\quad + 2\psi \left( G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \right) \tag{3.25}
\end{split}
\end{align*}
\]
This implies that
\[
\begin{align*}
\begin{split}
2\left( (a \cdot w_{n[k]+1}^x + b \cdot w_{n[k]+1}^y) + \left( (a \cdot w_{m[k]+1}^x + b \cdot w_{m[k]+1}^y) \right) \right) \\
&\geq 2\psi \left( G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \right) \tag{3.26}
\end{split}
\end{align*}
\]
From (3.20), the sequences \( \{ g(x_{n,k}) \}, \{ g(y_{n,k}) \}, \{ g(z_{n,k}) \} \) and \( \{ g(x_{n,k}) \}, \{ g(y_{n,k}) \}, \{ g(z_{n,k}) \} \) have subsequences converging to say \( \varepsilon_1 \) and \( \varepsilon_2 \) respectively, and \( \varepsilon_1 + \varepsilon_2 = \varepsilon > 0 \). By passing to subsequences, we may assume that 
\[
\lim_{k \to \infty} g(x_{n,k}) = \varepsilon_1 \quad \text{and} \quad \lim_{k \to \infty} g(y_{n,k}) = \varepsilon_2.
\]

Taking \( k \to \infty \) in (3.25) and using (3.26), we have
\[
0 = \lim_{k \to \infty} \left[ \left( a w_{x_{n,k}+1}^x + b w_{y_{n,k}+1}^y \right) \right] \\
\geq \lim_{k \to \infty} 2 \psi \left( g(x_{n,k}), g(y_{n,k}), g(z_{n,k}) \right) \\
> 0
\]
which is a contradiction. Therefore \( g(x_n) \) and \( g(y_n) \) are \( G \)-Cauchy sequences. By \( G \)-completeness of \( X \), there exists \( x, y \in X \) such that
\[
\lim_{n \to \infty} g(x_n) = x \quad \text{and} \quad \lim_{n \to \infty} g(y_n) = y.
\]

This together with the continuity of \( g \) implies that
\[
\lim_{n \to \infty} g(x_n) = g(x) \quad \text{and} \quad \lim_{n \to \infty} g(y_n) = g(y).
\]

Now suppose that the assumption (i) holds. From (3.2) and the commutativity of \( F \) and \( g \), we have
\[
g(x) = \lim_{n \to \infty} g(F(x_{n,k})), \quad g(y) = \lim_{n \to \infty} g(F(y_{n,k})), \quad F = \lim_{n \to \infty} F(g(x_n), g(y_n)).
\]

Similarly, we have
\[
g(y) = \lim_{n \to \infty} g(F(y_{n,k})), \quad g(x) = \lim_{n \to \infty} g(F(x_{n,k})), \quad F = \lim_{n \to \infty} F(g(y_n), g(x_n)).
\]

Hence, \( (x,y) \) is coupled coincidence point of \( F \) and \( g \).

Finally suppose that assumption (ii) holds. Since \( g(x_n) \) is non decreasing satisfying \( g(x_n) \to x \) and \( g(y_n) \) is non increasing satisfying \( g(y_n) \to y \), we have
\[
g(g(x_n)) \leq g(x) \quad \text{and} \quad g(g(y_n)) \geq g(y), \forall n \geq 0.
\]

Using the rectangle inequality and (3.1) we get
\[
a G(F(x,y), g(x), g(x)) + b G(F(y,x), g(y), g(y)) \leq
\]
\[
a G(F(x,y), g(x_{n,k}), g(x_{n,k})) + a G(g(x_{n,k}), g(x_{n,k})) \\
+ b G(F(y,x), g(y_{n,k}), g(y_{n,k})) + b G(g(y_{n,k}), g(y_{n,k}))
\]
\[
= a G(F(x,y), F(x_{n,k}), g(F(x_{n,k}))) + a G(g(x_{n,k}), g(x_{n,k})) \\
+ b G(F(y,x), F(y_{n,k}), g(F(y_{n,k}))) + b G(g(y_{n,k}), g(y_{n,k}))
\]
\[
\leq \frac{c}{2} \left( g(x_{n,k}) + g(y_{n,k}) \right) + \frac{d}{2} \left( g(x_{n,k}) + g(y_{n,k}) \right)
\]
\[
\leq \frac{c}{2} g(x_{n,k}) + \frac{d}{2} g(y_{n,k})
\]
\[
< 0
\]

Letting \( n \to \infty \) in the above inequality, we obtain
\[
G(F(x,y), g(x), g(x)) + G(F(y,x), g(y), g(y)) = 0
\]
which gives that \( F(F(x,y), g(x), g(x)) = G(F(y,x), g(y), g(y)) = 0 \), that is \( F(x,y) = g(x) \) and \( F(y,x) = g(y) \).

Therefore, \( (x,y) \) is a coupled coincidence point of \( F \) and \( g \). The proof of the theorem is complete.

**Corollary-20** Let \( (X, \leq) \) be a partially ordered set such that \( X \) is a complete \( G \)-metric space and \( F : X \times X \to X \) be a mapping having the mixed monotone property on \( X \). Suppose that there exists \( \psi \in \Psi \) and \( \phi \in \Phi \) such that
\[
M(x,y,z,u,v,w) \leq \phi \left( \frac{G(x,u,w) + G(y,v,z)}{2} \right) - 2\psi(G(x,u,w), G(y,v,z))
\]
where
\[
M(x,y,z,u,v,w) = a G(F(x,y), F(u,v), F(w,x)) + b G(F(y,x), F(v,u), F(z,w))
\]
for all \( a,b \in (0,\infty) \) and \( x, y, z, u, v, w \in X \) for which \( x \geq u \geq w \) and \( y \leq v \leq z \). If there exists \( x_0, y_0 \in X \) such that
\[
x_0 \leq F(x_0,y_0) \quad \text{and} \quad y_0 \geq F(y_0,x_0)
\]
and either
i. \( F \) is continuous or
ii. X has the following property:
   a. if a non decreasing sequence \( \{ x_n \} \) such that \( x_n \to x \) then \( x_n \leq x \) for all n,
   b. if a non increasing sequence \( \{ y_n \} \) such that \( y_n \to y \) then \( y_n \geq y \) for all n,
then F has a coupled fixed point in X, that is, \( x = F(x,y) \) and \( y = F(y,x) \).

Proof: Setting \( g(x) = x \) in Theorem-19, then the result follows.

**Theorem-21** Let \((X, \leq)\) be a partially ordered set such that X is a complete G- metric space and \( F: X \times X \to X \) and \( g: X \to X \) be mappings having the mixed g-monotone property on X. Suppose that there exists \( \psi \in \Psi \) such that

\[
M(x,y,z,u,v,w) \leq \frac{G(gx,gu,gw) + G(gy,gv,gv)}{2} - 2\psi(G(gx,gu,gw),G(gy,gv,gv))
\]

where

\[
M(x,y,z,u,v,w) = a G(F(x,y),F(u,v),F(w,z)) + b G(F(y,x),F(v,u),F(z,w))
\]

for all \( a,b \in (0, \infty) \) and \( x,y,z,u,v,w \in X \) for which \( gx \geq gu \geq gw \) and \( gy \leq gv \leq gz \). If there exists \( x_0,y_0 \in X \) such that

\[
gx_0 \leq F(x_0,y_0) \text{ and } gy_0 \geq F(y_0,x_0)
\]

and suppose \( F: (X \times X) \subseteq g(X) \), g is continuous and commutes with F, and also suppose either

i. F is continuous or
ii. X has the following property:
   a. if a non decreasing sequence \( \{ x_n \} \) such that \( x_n \to x \) then \( x_n \leq x \) for all n,
   b. if a non increasing sequence \( \{ y_n \} \) such that \( y_n \to y \) then \( y_n \geq y \) for all n,

then F and g have a coupled coincidence point, that is, there exists \( (x,y) \in X \times X \) such that

\[
g(x) = F(x,y) \text{ and } g(y) = F(y,x)
\]

Proof: It is sufficient if we take \( \phi(t) = t \) in Theorem-19 then the result follows.

**Theorem-22** Let \((X, \leq)\) be a partially ordered set such that X is a complete G- metric space and \( F: X \times X \to X \) and \( g: X \to X \) be mappings having the mixed g-monotone property on X. Suppose that there exists \( \psi \in \Psi \) and \( \phi \in \Phi \) such that

\[
M(x,y,z,u,v,w) \leq \phi \left( \frac{G(gx,gu,gw) + G(gy,gv,gv)}{2} \right) - 2\psi(G(gx,gu,gw),G(gy,gv,gv))
\]

where

\[
M(x,y,z,u,v,w) = a G(F(x,y),F(u,v),F(w,z)) + b G(F(y,x),F(v,u),F(z,w))
\]

for all \( a,b \in (0, \infty) \) and \( x,y,z,u,v,w \in X \) for which \( gx \geq gu \geq gw \) and \( gy \leq gv \leq gz \). If there exists \( x_0,y_0 \in X \) such that

\[
gx_0 \leq F(x_0,y_0) \text{ and } gy_0 \geq F(y_0,x_0)
\]

and suppose \( F: (X \times X) \subseteq g(X) \), g is continuous and commutes with F, and also suppose either

i. F is continuous or
ii. X has the following property:
   a. if a non decreasing sequence \( \{ x_n \} \) such that \( x_n \to x \) then \( x_n \leq x \) for all n,
   b. if a non increasing sequence \( \{ y_n \} \) such that \( y_n \to y \) then \( y_n \geq y \) for all n,

then F and g have a coupled coincidence point, that is, there exists \( (x,y) \in X \times X \) such that

\[
g(x) = F(x,y) \text{ and } g(y) = F(y,x)
\]

Proof: It is sufficient if we take \( \psi(t) = max \{ t_1, t_2 \} \) in Theorem-19, we get the above result.

**Theorem-23** Let \((X, \leq)\) be a partially ordered set such that X is a complete G- metric space and \( F: X \times X \to X \) and \( g: X \to X \) be mappings having the mixed g-monotone property on X. Suppose that there exists \( \psi \in \Psi \) and \( \phi \in \Phi \) such that

\[
M(x,y,z,u,v,w) \leq \phi \left( \frac{G(gx,gu,gw) + G(gy,gv,gv)}{2} \right) - 2\psi(G(gx,gu,gw),G(gy,gv,gv))
\]

Proof: Setting \( g(x) = x \) in Theorem-19, then the result follows.
where
\[ M(x, y, z, u, v, w) = aG(F(x, y), F(u, v), F(w, z)) + bG(F(y, x), F(v, u), F(z, w)) \]
for all \( a, b \in (0, \infty) \) and \( x, y, z, u, v, w \in X \) for which \( gx \geq gu \geq gw \) and \( gy \leq gv \leq gz \). If there exists \( x_0, y_0 \in X \) such that
\[ gx_0 \leq F(x_0, y_0) \quad \text{and} \quad gy_0 \geq F(y_0, x_0) \]
and suppose \( F : (X \times X) \subseteq g(X) \), \( g \) is continuous and commutes with \( F \), and also suppose either
1. \( F \) is continuous or
2. \( X \) has the following property:
   a. if a non decreasing sequence \( \{x_n\} \) such that \( x_n \to x \) then \( x_n \leq x \) for all \( n \)
   b. if a non increasing sequence \( \{y_n\} \) such that \( y_n \to y \) then \( y_n \geq y \) for all \( n \)
then \( F \) and \( g \) have a coupled coincidence point, that is, there exists \( (x, y) \in X \times X \) such that
\[ g(x) = F(x, y) \quad \text{and} \quad g(y) = F(y, x). \]

**Proof:** In Theorem – 19, taking \( \psi(t_1, t_2) = \psi(t_1 + t_2) \) for all \( (t_1, t_2) \in [0, \infty)^2 \) we get the desired results.

**Theorem 24** In addition to the hypothesis of Theorem – 19, suppose that for all \( (x, y), (x', y') \in X \times X \), there exists \( (u, v) \in X \times X \) such that \( (F(u, v), F(v, u)) \) is comparable with \( (F(x, y), F(y, x)) \) and \( (F(x', y'), F(y', x')) \). Then \( F \) and \( g \) have a unique coupled common fixed point.

**Proof:** From Theorem 19, the set of coupled coincidence is non empty. Assume that \((x, y)\) and \((x', y')\) are coupled coincidence points of \( F \) and \( g \). We shall show that
\[ g(x) = g(x') \quad \text{and} \quad g(y) = g(y') \] (3.33)

By assumption, there exists \( (u, v) \in X \times X \) such that \( (F(u, v), F(v, u)) \) is comparable with \( (F(x, y), F(y, x)) \) and \( (F(x', y'), F(y', x')) \). Putting \( u_0 = u, v_0 = v \) and choosing \( u_1, v_1 \in X \) such that
\[ g(u_1) = F(u_0, v_0) \quad \text{and} \quad g(v_1) = F(v_0, u_0). \]

Then, similarly as in the proof of Theorem 19, we can inductively define sequences \( \{g(u_n)\} \) and \( \{g(v_n)\} \) in \( X \) by
\[ g(u_{n+1}) = F(u_n, v_n) \quad \text{and} \quad g(v_{n+1}) = F(v_n, u_n), \quad \forall \ n \geq 0 \]
Since \( (F(x', y'), F(y', x')) \) is comparable with \( (g(x'), g(y')) \) and \( (F(u, v), F(v, u)) = (g(u_1), g(v_1)) \) are comparable, so without loss of generality, we may assume that
\[ (F(x', y'), F(y', x')) = (g(x'), g(y')) \leq (F(u, v), F(v, u)) = (g(u_1), g(v_1)) \]
and
\[ (F(x, y), F(y, x)) = (g(x), g(y)) \leq (F(u, v), F(v, u)) = (g(u_1), g(v_1)) \]
This means that
\[ g(x') \leq g(u_1) \quad \text{and} \quad g(y') \geq g(v_1) \]
and
\[ g(x) \leq g(u_1) \quad \text{and} \quad g(y) \geq g(v_1) \]
Using the fact that \( F \) is a mixed g-monotone mapping, we can inductively show that
\[ g(x') \leq g(u_n) \quad \text{and} \quad g(y') \geq g(v_n), \quad \forall \ n \geq 1 \]
and
\[ g(x) \leq g(u_n) \quad \text{and} \quad g(y) \geq g(v_n), \quad \forall \ n \geq 1 \]
Thus from (3.1), we get
\[
\begin{align*}
& aG(g(u_{(n+1)}), g(x), g(x)) + bG(g(v_{(n+1)}), g(y), g(y)) \\
& = aG(F(u_n, v_n), F(x, y), F(x, y)) + bG(F(v_n, u_n), F(y, x), F(y, x)) \\
& \leq \phi\left(\left(\frac{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))}{2}\right)^2 - 2 \psi\left(\frac{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))}{2}\right) + \phi\left(\left(\frac{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))}{2}\right)^2\right)\right)
\end{align*}
\]
which implies that
\[
\begin{align*}
& aG(g(u_{(n+1)}), g(x), g(x)) + bG(g(v_{(n+1)}), g(y), g(y)) \leq \psi\left(\frac{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))}{2}\right) + \phi\left(\left(\frac{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))}{2}\right)^2\right)
\end{align*}
\]
that is the sequences \([ G(g(u_n),g(x),g(x)) + G(g(v_n),g(y),g(y)) \]) is decreasing. Therefore there exists \( \delta \geq 0 \) such that
\[
\lim_{n \to \infty} [ G(g(u_n),g(x),g(x)) + G(g(v_n),g(y),g(y)) ] = \delta
\]

We shall show that \( \delta = 0 \) suppose to the contrary that \( \delta > 0 \). Therefore, \( G(g(u_n),g(x),g(x)) \) and \( G(g(v_n),g(y),g(y)) \) have subsequences converging to \( \delta_1, \delta_2 \) respectively, with
\[
\delta_1 + \delta_2 = \delta > 0
\]

Taking the limit up to subsequences as \( n \to \infty \) in (3.34) we have
\[
\delta \leq \delta - 2\lim_{n \to \infty} \psi \left[ G(g(u_n),g(x),g(x)) + G(g(v_n),g(y),g(y)) \right] < \delta
\]
which is a contradiction. Thus \( \delta = 0 \), that is
\[
\lim_{n \to \infty} \left( G(g(u_n),g(x),g(x)) + G(g(v_n),g(y),g(y)) \right) = 0
\]
which implies that
\[
\lim_{n \to \infty} G(g(u_n),g(x),g(x)) = \lim_{n \to \infty} G(g(v_n),g(y),g(y)) = 0 \quad (3.35)
\]

Similarly, one can show that
\[
\lim_{n \to \infty} G(g(u_n),g(x^*),g(x^*)) = \lim_{n \to \infty} G(g(v_n),g(y^*),g(y^*)) = 0 \quad (3.36)
\]

Therefore, from (3.35), (3.36) and the uniqueness of the limit, we get \( g(x) = g(x^*) \) and \( g(y) = g(y^*) \). So (3.33) holds. Since \( g(x) = F(x,y) \) and \( g(y) = F(y,x) \), by commutativity of \( F \) and \( g \) we have
\[
g(g(x)) = g(F(x,y)) = F(g(x),g(y)) \quad \text{and} \quad g(g(y)) = g(F(y,x)) = F(g(y),g(x)) \quad (3.37)
\]

Denote \( g(x) = z \) and \( g(y) = w \), then by (3.37) we get
\[
g(x) = F(x,w) \quad \text{and} \quad g(w) = F(w,x)
\]
(3.38)

Thus \( (x,w) \) is a coincidence point. Then form (3.33) with \( x^* = z \) and \( y^* = w \), we have \( g(x) = g(x) \) and \( g(y) = g(w) \), that is
\[
g(z) = z \quad \text{and} \quad g(w) = w
\]
(3.39)

From (3.38) and (3.39) we get
\[
g(x) = F(x,w) = z \quad \text{and} \quad g(w) = F(w,z) = w
\]
(3.40)

Then \( (x,w) \) is a coupled common fixed point of \( F \) and \( g \). To prove the uniqueness, assume that \( (p,q) \) is another coupled fixed point. Then by (3.33) we have
\[
g(z) = g(p) = z \quad \text{and} \quad g(w) = g(q) = q
\]
(3.41)

This complete the proof of the Theorem.

Remark-25 Some special cases of Theorem 19 yields existing results as detailed below.

1. In Theorem 19, if we take following conditions then we get existing results:
   i. If we take \( a = 1, b = 0, \phi(t) = kt \) where \( k \in (0,1) \) \( g = I_k \) (identity mapping) and \( \psi(t_1,t_2) = 0 \) then we get the result of Choudhury and Maity [6].
   ii. If we take \( a = 1, b = 0 \) and \( \psi(t_1,t_2) = 0 \) then we get the result of Aydi et al. [3].
   iii. If we take \( a = 1, b = 1, \phi(t) = 2kt \) for \( k \in (0,\frac{1}{2}) \) and \( \psi(t_1,t_2) = 0 \) then we get the result of Nashine [16].
   iv. If we take \( a = 1, b = 1, \phi(t) = 2kt \) for \( k \in [0,1) \) and \( \psi(t_1,t_2) = 0 \) then we get the result of Karapinar et al. [7].
   v. If we take \( a = 1, b = 0, \phi(t) = 2t \) then we get the result of Wangkeeree and Bantaojai [18].
   vi. If we take \( a = 1, b = 0, g = I_k \) (identity mapping) \( \phi(t) = t \) then we get the result of Luong and Thuan [10].

Example-26 Let \( X = \mathbb{R} \). Define \( G: X \times X \times X \to [0, \infty) \) by
\[
G(x,y,z) = | x - y | + | y - z | + | z - x |
\]
\[
F(x,y) = 2x - 3y, \quad g(x) = x
\]
also \( a = 2, b = 2, \phi (t) = 6t \) and \( \psi (r_1, r_2) = \frac{r_1 + r_2}{5} \). Then (3.1) indicates that \((0,0)\) is a fixed point.

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**REFERENCE**

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