Lattice Points on the Homogeneous Cubic Equation with Four Unknowns \((x + y)(xy + w^2) = (k^2 - 1)z^3, k > 1\)

Gopalan MA¹, Sangeetha V², Manju Somanath³

1. Professor, Department of Mathematics, Srimathi Indhira Gandhi College, Trichy-2, Tamil Nadu, India
2. Assistant Professor, Department of Mathematics, National College, Trichy-1, Tamil Nadu, India
3. Assistant Professor, Department of Mathematics, National College, Trichy-1, Tamil Nadu, India

*Corresponding Author: Professor, Department of Mathematics, Srimathi Indhira Gandhi College, Trichy, Tamil Nadu, India, E.mail: mayilgopalan@gmail.com, Mobile: 9944848938

Received 24 December; accepted 27 January; published online 01 February; printed 16 February 2013

ABSTRACT

The homogeneous cubic equation with four unknowns represented by \((x + y)(xy + w^2) = (k^2 - 1)z^3, k > 1\), is analyzed for its patterns of non zero distinct integral solutions. Three different patterns of solutions are presented. A few interesting relations between the solutions and special numbers, namely, polygonal numbers, centered polygonal numbers, star numbers and nasty numbers are exhibited.

Key words: Homogeneous cubic, Lattice Points, Integral solutions.

MSC Classification Number: 11D09.

1. INTRODUCTION

The Cubic Equation offers an unlimited field for research because of their variety (Dickson 1952, Mordell 1969). In particular, one may refer (Gopalan et al, 2009, 2010, 2011, 2012) for cubic equation with four unknowns. This communication concerns with yet another interesting equation \((x + y)(xy + w^2) = 2(k^2 - 1)z^3\) representing a homogeneous cubic with four unknowns for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

2. NOTATION

\(t_{m,n}\) = Polygonal number of rank \(n\) with sides \(m\).
\(c_{p,2n}\) = Centered icositetragonal number.
\(c_{p,3n}\) = Centered dodecagonal number.
\(c_{p,4n}\) = Centered octagonal number.
\(c_{p,6n}\) = Centered nonagonal number.
\(c_{p,8n}\) = Centered square number.
\(S_n\) = Star number.

3. METHOD OF ANALYSIS

The homogeneous cubic represented by the cubic equation is

\[(x + y)(xy + w^2) = 2(k^2 - 1)z^3, \quad (1)\]

It is observed that (1) is satisfied by infinitely many non-zero distinct integral solutions. For the sake of clear understanding, we present below different patterns of solutions to (1).

3.1. Pattern - 1

Introducing the linear transformations

\[x = u + v; \quad y = u - v; \quad z = u\]

in (1), it is written as

\[w^2 = (k^2 - 2)u^2 + v^2, \quad (3)\]

which is satisfied by

\[u = 2rs; \quad v = (k^2 - 2)r^2 - s^2; \quad w = (k^2 - 2)r^2 + s^2 \quad (4)\]

From (4) and (2), the non-zero integral solutions of (1) are given by

\[x = 2rs + (k^2 - 2)r^2 - s^2;\]
\[y = 2rs - (k^2 - 2)r^2 + s^2;\]
\[z = 2rs;\]
\[w = (k^2 - 2)r^2 - s^2.\]


3.2. Pattern - 2

Rewrite (3) as  
\[(k^2 - 2)u^2 + v^2 = w^2 + 1\]  
Assume \(w = a^2 + (k^2 - 2)b^2\)  
(5)

Write 1 as  
\[1 = \frac{(1 + 2\sqrt{k^2 - 2})/2}{(k^2 - 2)^{1/2}}\]  
(6)

Using (6) and (7) in (5) and applying the method of factorization, define  
\[v + i\sqrt{(k^2 - 2)}u = (a + i\sqrt{(k^2 - 2)}b)^2 \frac{(1 + 2\sqrt{k^2 - 2})/2}{(k^2 - 2)^{1/2}}\]  
(7)

Let \(k = 4a - 2\).

Equating the real and imaginary parts, we get  
\[u = \frac{a^2 + 2ab - (16a^2 - 16a + 2) + 2a^2}{2}\] \(v = a^2 - 2ab + (16a^2 - 16a + 2) - (16a^2 - 16a + 2)\] \(w = a^2 + (16a^2 - 16a + 2)b^2\)  
(8)

As our interest is on finding integer solution, it is observed that it is possible to choose \(a, a, b\) so that \(u, v\) and \(w\) are integers and thus in view of (2) one obtains non-zero distinct solutions to (1).

For illustration, \(a = 1\) in (8) we obtain,  
\[u = \frac{2a^2 + 4ab + 2b^2}{2}, \quad v = \frac{2a^2 - 4ab + 2b^2}{2}, \quad w = a^2 + 2b^2\]  
(9)

Let \(a = 3A, b = 3B\) \((A \neq B)\), then  
\[u = 18A^2 + 18AB - 36B^2\] \(v = 3A^2 - 24AB - 6B^2\] \(w = 9A^2 + 18B^2\)  
Using (9) in (2), the integral solutions to (1) is obtained as  
\[x = 9A^2 - 18AB - 18B^2; \quad y = 3A^2 + 30AB - 6B^2;\] \[z = 6A^2 + 6AB - 12B^2; \quad w = 9A^2 + 18B^2\]  
A few properties of the above solutions are
1. \(z(A, 1) - 12t_{2a, a} = -12\).
2. \(x(A, 1) + y(A, 1) + 25\) is a centered icositetragonal number.
3. \(3w(1, B) - 27\) is a Nasty number.
4. \(2x(A, 1) + 6z(A, 1) + 162\) is a Nasty number.
5. \(a(1, B) + w(1, B) - 14\) is a centered dodecagonal number.
6. \(y(1, B + 1) + z(1, B + 1) + 3T_n - 36G_9 = 12\)

3.3 Pattern - 3

Rewrite (3) as  
\[1 + v^2 = w^2 - (k^2 - 2)u^2\]  
(10)

Assume \(v = a^2 - (k^2 - 2)b^2\)  
(11)

Write 1 as  
\[1 = \frac{(a^2 - 1) + \sqrt{k^2 - 2}}{2}\]  
(12)

Using (11) and (12) in (10) and applying the method of factorization, define  
\[w + \sqrt{k^2 - 2}u = (a + \sqrt{k^2 - 2}b)^2 \frac{(a^2 - 1) + \sqrt{k^2 - 2}}{2}\]  
(13)

Equating the rational and irrational parts, we obtain  
\[u = \frac{a^2 + 2ab - (k^2 - 2)}{2}\] \(w = a^2 - (k^2 - 2)b^2\)  
Let \(k = a + 1\), then  
\[u = 2ab(a^2 + 2a) + 2a^2 + 2(a^2 + 2a - 1)b^2\] \[w = a^2 + 2ab - 2\] \[v = a^2 - (a^2 + 2a - 1)b^2\]  
As our interest is on finding integer solution, it is observed that it is possible to choose \(a, a, b\) so that \(u, v\) and \(w\) are integers and thus in view of (2) one obtains non-zero distinct solutions to (1).

For illustration, let \(a = 1\) in (13) we obtain,  
\[w = 3a^2 + 6b^2 + 8ab\] \(u = 2a^2 + 4b^2 + 6ab\] \(v = a^2 - 2b^2\]  
(14)

Using (14) in (2), we get,
Some interesting properties of the above solutions are:

1. \(x(1, b) - y(1, b) + 4t_{ab} = 2\).
2. \(w(1, b) - z(1, b)\) is a centered square number.
3. \(2w(1, b) - y(1, b) - 4t_{a+b} + 4t_{ab} + 12\) is a Star Number.
4. \(x(a, 1) + w(a, 1) - 12f_{a} - f_{a+1} + f_{a-1} = 0\)
5. \(y(2, b(b + 1)) - x(1, b(b + 1)) - 3p_{b(b+1)} = 4f_{n}\).
6. \(2z(A + 1, 1) + w(A + 1, 1) - 2q_{a(A+1)} - 22f_{ab} = 8\).
7. \(x(1, b) - 3y(1, b) + 24r_{ab} + 4t_{ab} = 0\).

4. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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