The ternary quadratic equation $z^2 = a^2(x^2 + y^2) + bxy$ is analysed for its non-zero distinct integral points on it. A few interesting properties among the solutions are presented.

**Keywords:** Ternary quadratic, homogeneous cone, integral points.

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**Notation:**
- $t_{m,n}$ = Polygonal number of rank $n$ with sides $m$
- $p_{m,n}$ = Pyramidal number of rank $n$ with sides $m$
- $p_n$ = Pronic number
- $g_n$ = Gnomonic number
- $SO_n$ = Stella octangular number

1. **INTRODUCTION**

The ternary quadratic Diophantine equation offers an unlimited field for research because of their variety (Dickson L.E 1952, Mordell L.J 1969). For an extensive review of various problems one may refer (Gopalan M.A, et al 2008, 2010, 2011). In this context one may also see (Gopalan M.A, et al 2011, 2012). This communication concerns with yet another interesting ternary quadratic equation $z^2 = a^2(x^2 + y^2) + bxy$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. **METHOD OF ANALYSIS**

Consider the equation

$$z^2 = a^2(x^2 + y^2) + bxy \tag{1}$$

Introducing the linear transformations $x = u + v, y = u - v \tag{2}$ in (1), it is written as

$$z^2 = u^2 \left[2a^2 + b \right] + v^2 \left[2a^2 - b \right] \tag{3}$$

Again, the assumption $z = 2aw, u = X + \left(2a^2 - b\right)T, v = X - \left(2a^2 + b\right)T \tag{4}$ in (3), leads to

$$w^2 - X^2 = \left(2a^2 - b\right) \left(2a^2 + b\right)T^2 \tag{5}$$

We present below different choices of solutions to (5) and hence obtain different patterns of solutions to (1).
2.1. Pattern: 1

Equation (5) is equivalent to the following system of equations

\[ w + X = \left(2a^2 + b\right)T^2 \]  
\[ w - X = \left(2a^2 - b\right) \]  

from which, on solving we get

\[ w = \frac{1}{2} \left[ \left(2a^2 + b\right)T^2 + \left(2a^2 - b\right) \right] \]  
\[ X = \frac{1}{2} \left[ \left(2a^2 + b\right)T^2 - \left(2a^2 - b\right) \right] \]

Substituting the values of \(w\) and \(X\) in (4) and using (2), the non-zero distinct integral values of \(x, y, z\) satisfying (1) are given by

\[ x = x(a, b, T) = \left(2a^2 + b\right)T^2 - \left(2a^2 - b\right) - 2bT \]  
\[ y = y(a, T) = 4a^2T \]  
\[ z = z(a, b, T) = a \left[ \left(2a^2 + b\right)T^2 + \left(2a^2 - b\right) \right] \]

Properties:

1. \(z(a, 1, 1) + y(a, 1) = 8p_5^a\)
2. \(y(a, a) - x(a, a, 2) - t_{10, a} + 2t_{5, a} \equiv 0 \pmod{2}\)
3. \(y(a, a + 1) = 8p_5^a\)
4. \(x(1, a, a) + y(1, a, 1) + z(1, a, 1) - 6p_7^a - 2y_{3, a} - g_a = -1\)
5. \(y(a - 1, 1) + x(1, 1, a - 1) - t_{16, a} + 5g_a = 3\)

2.2 Pattern: 2

Write (5) as

\[ w^2 = \left(4a^4 - b^2\right)T^2 + X^2 \] (10)

Choose \(a\) and \(b\) such that \(4a^4 - b^2\) is square free. For this case, the values of \(w, T\) and \(X\) satisfying (10) are given by

\[ w = \left(4a^4 - b^2\right)p^2 + q^2 \]  
\[ T = 2pq \]  
\[ X = \left(4a^4 - b^2\right)p^2 - q^2 \]

Following the procedure presented in pattern 1, the non-zero distinct integral values of \(x, y, z\) satisfying (1) are obtained as

\[ x = x(a, b, p, q) = 8a^2p^2 - 2b^2p^2 - 2q^2 - 4bpq \]  
\[ y = y(a, p, q) = 8a^2pq \]  
\[ z = z(a, b, p, q) = 8a^3p^2 - 2ab^2p^2 + 2aq^2 \]

Properties:

1. \(x(a, 1, 1, 1) - S_a \equiv -3 \pmod{2}\)
2. \(x(b - 2, 1, 1, 1) - y(b - 2, 1, 1, 1) = -8\)
3. \(z(1, 1, p + 2, p + 2) + y(1, p + 2, p + 2) - t_{34, p} = 64 \pmod{79}\)
2.3. Pattern: 3

Consider (5) as

\[ w^2 + \left( b^2 - 4a^4 \right) T^2 = X^2 \]  

(11)

Assume

\[ X = M^2 + \left( b^2 - 4a^4 \right) N^2, M, N \neq 0 \]  

(12)

Write 1 as

\[ 1 = \frac{\left( 2a^2 + i\sqrt{b^2 - 4a^4} \right) \left( 2a^2 - i\sqrt{b^2 - 4a^4} \right)}{b^2} \]  

(13)

Substitute (12) and (13) in (11). Applying the method of factorization and performing a few calculations the integral values of \( w, X \) and \( T \) are represented by

\[ w = 4a^2 s^2 b - 2a^2 t^2 b \left( b^2 - 4a^4 \right) - 2stb \left( b^2 - 4a^4 \right) \]

\[ T = 4a^2 stb + s^2 b - \left( b^2 - 4a^4 \right) t^2 b \]

\[ X = s^2 b^2 + t^2 b^4 - 4a^4 t^2 b^2 \]

Substitute the values of \( w \) and \( T \) in (4) and using (2) the non-zero distinct integral values of \( x, y \) and \( z \) are found to be

\[ x = x(a,b,s,t) = 4t^2 b^4 - 16a^4 t^2 b^2 - 8a^2 stb^2 \]

\[ y = y(a,b,s,t) = 16a^4 stb + 4a^2 s^2 b - 4a^2 t^2 b^3 + 16a^4 s^2 b \]

\[ z = z(a,b,s,t) = 2a \left( 2a^2 s^2 b - 2a^2 t^2 b^3 + 8a^4 t^2 b^2 - 2stb^3 + 8a^4 stb \right) \]

Properties:

1. \( x(1,1,1,s) + 4g_s = -16 \)
2. \( x(1,1,1,s) - y(1,1,1,s) - 4p_s \equiv 0 \text{ (mod 4)} \)
3. \( y(1,1,t,1) - z(1,1,t,1) - t_{30,t} - 10p_t \equiv 8 \text{ (mod 31)} \)
4. \( y(1,1,t,1) + z(1,1,t,1) + 6S_3 b \equiv 0 \text{ (mod 66)} \)
5. \( x(1,1,t,1) + y(1,1,t,1) + z(1,1,t,1) - t_{26,t} \equiv 8 \text{ (mod 31)} \)

2.4. Pattern: 4

Rewrite (6) as

\[ w^2 - \left( 4a^4 - b^2 \right) T^2 = X^2 \]  

(14)

Assume \( X = u^2 - \left( 4a^4 - b^2 \right) v^2, u, v \neq 0 \)  

(15)

Substitute (15) in (14) and employing the method of factorization, we get

\[ w = u^2 + \left( 4a^4 - b^2 \right) v^2 \]

\[ T = 2uv \]

Substitute \( w \) and \( T \) in (4) and using (2) the non-zero distinct integral values of \( x, y \) and \( z \) satisfy (1) are given by

\[ x = x(a,b,u,v) = 2u^2 - 8a^4 v^2 + 2b^2 v^2 - 4buv \]

\[ y = y(a,u,v) = 8a^2 uv \]

\[ z = z(a,b,u,v) = 2au^2 + 8a^5 v^2 - 2ab^2 v^2 \]

Properties:

1. \( x(1,1,1,v) + z(1,1,1,v) + 2g_v = 2 \)
On the ternary quadratic equation \( z^2 = a^2 (x^2 + y^2) + bxy \),

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http://www.discovery.org.in/ijjs.htm

2. \( y(1,1,v) + z(1,1,1,v) - 6p_v - g_v = 3 \)
3. \( x(1,1,u,1) + z(1,1,u,1) - t_{10,u} \equiv 0 \pmod{1} \)
4. \( \frac{x(1,1,u-1,1) + z(1,1,u-1,1) - 4u}{y(1,u-1,1)} = -1 \)
5. \( x(a,a,1,1) + y(a,1,1) + 16t_{3,3,a}^2 - 2p_a - g_a = 0 \pmod{3} \)

3. CONCLUSION
One may search for other patterns of solution and their corresponding properties.

REFERENCES