Integral points on the hyperbola \((a+2)x^2 - ay^2 = 4a(k-1) + 2k^2\), \(a, k > 0\)

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ABSTRACT

Integral points on a class of hyperbolas represented by the binary quadratic equation, \((a+2)x^2 - ay^2 = 4a(k-1) + 2k^2\), \(a, k > 0\) are obtained. A few interesting relations among the solutions are presented.

Keywords: Binary quadratic, hyperbolic, integral points.

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1. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-3]. In [4-7, 9-14] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. However, in [8] it is shown that the hyperbola given by \(3x^2 + xy = 14\) has only finite number of integral points. In [15], the recurrence relations satisfied by the solutions of \(x^2 + 6xy + y^2 + 40x + 8y + 40 = 0\) is given. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting hyperbola given by \((a + 2)x^2 - ay^2 = 4a(k-1) + 2k^2\). The recurrence relations satisfied by the solutions \(x\) and \(y\) are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

Consider the class of hyperbolas represented by the binary quadratic equation

\[(a+2)x^2 - ay^2 = 4a(k-1) + 2k^2\]  

(1)

The substitution of the linear transformations 

\[x = X + aT, \quad y = X + (a + 2) T\]  

(2)

in (1) leads to

\[X^2 = a(a + 2)T^2 + 2a(k-1) + k^2\]

which is satisfied by

\[T_{n+1} = (a + k)\tilde{T}_n + \tilde{X}_n\]

\[X_{n+1} = (a + k)\tilde{X}_n + (a^2 + 2a)\tilde{T}_n\]

(3)

where

\[\tilde{T}_n = \frac{f}{2}, \quad \tilde{X}_n = \frac{g}{2\sqrt{a^2 + 2a}}\]

\[f = (a + 1 + \sqrt{a^2 + 2a})^{n+1} + (a + 1 - \sqrt{a^2 + 2a})^{n+1}\]

\[g = (a + 1 + \sqrt{a^2 + 2a})^{n+1} - (a + 1 - \sqrt{a^2 + 2a})^{n+1}\]

(4)

(5)

From (2) and (3), the non-zero distinct integral points on the hyperbola (1) are given by

\[x_{n+1} = \left(\frac{2a+k}{2}\right)f + \left(\frac{2a^2 + 2a + ak}{2\sqrt{a^2 + 2a}}\right)g\]

\[y_{n+1} = \left(\frac{2a+k+2}{2}\right)f + \left(\frac{2a^2 + 4a + ak + 2k}{2\sqrt{a^2 + 2a}}\right)g\]

where \(n = -1, 0, 1, \ldots\)
RESEARCH

The recurrence relations satisfied by \( x_{n+1}, y_{n+1} \) are respectively as follows:

\[
\begin{align*}
\quad \quad x_{n+3} - 2(a + 1)x_{n+2} + x_{n+1} &= 0, \quad x_0 = 2a + k, \quad x_1 = 4a^2 + 2ak + 4a + k\\
y_{n+3} - 2(a + 1)y_{n+2} + y_{n+1} &= 0, \quad y_0 = 2a + k + 2, \quad y_1 = 4a^2 + 2ak + 8a + 3k + 2
\end{align*}
\]

A few interesting properties are tested among the solutions of (1) are presented below:

(i) \( (2a^2 + ak + 4a + 2k)x_{2n+2} - (2a^2 + 2a + 4k)y_{2n+2} \equiv 0 \mod (k^2 + 2ak - 2a), k > 1 \)

(ii) \( [(2a^2 + ak + 4a + 2k)x_{n+3} - (2a^2 + 2a + ak)y_{n+1}^2] - (a^2 + 2a)(2a + k + 2)x_{n+3} - (2a + k + 2)y_{n+1}^2 \equiv 0 \mod (k^2 + 2ak - 2a), k > 1 \)

(iii) \( (2a^2 + ak + 4a + 2k)x_{3n+3} - (2a^2 + 2a + ak)y_{3n+3}^2 \equiv 0 \mod (k^2 + 2ak - 2a), k > 1 \)

(iv) \( (2a^2 + ak + 4a + 2k)x_{2n+2} - (2a^2 + 2a + ak)y_{2n+2} \pm 2(k^2 + 2ak - 2a) \) is a perfect square

NOTE

One may get a different solution pattern to (1) by considering the linear transformation \( x = X - a T, y = Y - (a^2) T \).

3. GENERATION OF SOLUTIONS

We present below a general formula for generating a sequence of solutions to (1) being given its initial solution.

The initial solutions to (1) is given by \( x_0 = k, \quad y_0 = k - 2 \)

Let \( x_1 = h - x_0, \quad y_1 = y_0 + h, \quad h \in \mathbb{Z} - \{0\} \)

be the second solution of (1). Substitution of (6) in (1) gives

\[
\begin{align*}
x_1 &= a + 1 \quad a \neq b \quad a + 2 \quad a + 4 \end{align*}
\]

Thus, we have the second solution to (1) written in the matrix form as

\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a+1 & a \\ a+2 & a+4 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\]

The repetition of the above process leads to the general solution \( (x_{n+1}, y_{n+1}) \) to (1) as

\[
\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \vec{X} \\ \vec{Y} \end{bmatrix} a \vec{T}_n = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\]

Putting \( n = 0,1,2, \ldots \) in (7), one can generate sequence of values of \( x \) and \( y \) based on the given solution \( (x_0, y_0) \).

4. CONCLUSION

One may search for other patterns of solutions and their corresponding properties.

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