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On the ternary biquadratic non-homogeneous equation, (2k + 1)(x2 + y2 + xy) = z4

Gopalan MA, Vidhyalakshmi S, Sumathi G

1. Department of Mathematics, Shrimathi Indira Gandhi College, Trichy-600002, Tamil Nadu, India, E-mail: mayilgopalan@gmail.com
2. Department of Mathematics, Shrimathi Indira Gandhi College, Trichy-600002, Tamil Nadu, India, E-mail: vidhyasigc@gmail.com
3. Department of Mathematics, Shrimathi Indira Gandhi College, Trichy-600002, Tamil Nadu, India, E-mail: b.deepacharan@gmail.com

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ABSTRACT

The ternary biquadratic non-homogeneous equation represented by the diophantine equation

$$(2k+1)(x^2+y^2+xy)=z^4$$

is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

Keywords: Integral solutions,ternary biquadratic non-homogeneous equation ,lattice points.

M.Sc 2000 mathematics subject classification: 11D25

NOTATIONS:

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

S_n : Star number of rank n

$\Pr_n$: Pronic number of rank n

So_n : Stella octangular number of rank n

j_n : Jacobsthal number of rank n

J_n : Jacobsthal lucas number of rank n

Ky_n : Kynea number of rank n

1. INTRODUCTION

The biquadratic diophantine (homogeneous or non-homogeneous) equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-11] for ternary non-homogeneous biquadratic equations. This communication concerns with yet another interesting ternary non-homogeneous biquadratic equation given by

$$(2k+1)(x^2+y^2+xy)=z^4$$

for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS

The diophantine equation representing a non-homogeneous biquadratic equation is

$$(2k+1)(x^2+y^2+xy)=z^4 \tag{1}$$

Introducing the linear transformations,

$$x=(2k+1)(u+v),\; y=(2k+1)(u-v),\; z=(2k+1)w \tag{2}$$

in (1),it leads to

$$3u^2+v^2=(2k+1)w^4 \tag{3}$$

Gopalan et al.
On the ternary biquadratic non-homogeneous equation, (2k + 1)(x2 + y2 + xy) = z4,
Indian Journal of Engineering, 2012, 1(1), 37-40,

Assume $w = w(a,b) = b^2 + 3a^2$, $a,b \neq 0$ (4)

To find the solution for (3), choose k such that

$$(2k+1) = B^2 + 3A^2 \quad \text{, where } B = 2p+1, A = 2q \tag{5}$$

Employing the method of factorization, define

$$(v + i\sqrt{3}u) = \left((2p+1) + i(\sqrt{3}2q)\right)(b + i\sqrt{3}a)^4 \tag{6}$$

Equating real and imaginary parts in (6) we get,

$$v(a,b,p,q) = (2p+1)(b^4 + 9a^4 - 18a^2b^2) - 6q(4ab^3 - 12ab) \tag{7}$$

$$u(a,b,p,q) = (2p+1)(4ab^3 - 12ab) + 2q(b^4 + 9a^4 - 18a^2b^2) \tag{8}$$

Substituting (7) and (8) in (2), the corresponding integral solutions of (1) are presented by

$$x = x(a,b,p,q) = (2k+1)\begin{pmatrix} (2p+1)(4ab^3 - 12ab + b^4 + 9a^4 - 18a^2b^2) \\ + 2q(b^4 + 9a^4 - 18a^2b^2 - 12ab^3 + 36ab) \end{pmatrix}$$

$$y = y(a,b,p,q) = (2k+1)\begin{pmatrix} (2p+1)(4ab^3 - 12ab - b^4 - 9a^4 + 18a^2b^2) \\ + 2q(b^4 + 9a^4 - 18a^2b^2 + 12ab^3 - 36ab) \end{pmatrix}$$

$$z = z(a,b,p,q) = \left((2p+1)^2 + 3(2q)^2\right)(b^2 + 3a^2)$$

2.1. Properties

1. $x(1,b,p,q) + y(1,b,p,q) - 4(2k+1)(2p+1)So_b - (2k+1)(2q)(2t_{4,b}^2 - 36t_{4,b} + 18) \equiv 0 (\mathrm{mod}\, 20)$

2. $x(a,1,p,q) - y(a,1,p,q) - (2k+1)(2p+1)(2S_a + 24t_{3,a} - 60t_{4,a} + 18t_{4,a}^2) \equiv 0 (\mathrm{mod}\, 48)$

3. Each of the following is a nasty number.

(i) $6z(a, a, 6k^2 + 6k - 2l^2 + 1, 2l(2k+1))$

(ii) $6z(a, a, 6l^2 - 2k^2 - 2k - 1, 2l(2k+1))$, When p and q are different pairty.

4. $y(1,b,p,q) - (2k+1)(2p+1)(36P_b^3 - 4t_{3,b}^2 + (b-12)^2 - 153) -$

$$(2k+1)(2q)(4_{3,b}^2 + P_b^6 + 3So_b - 32\,\mathrm{Pr}_b - 10t_{4,b} + 9) \equiv 0$$

It is observed that, in (5), one may also take $B = 2p, A = 2q+1$ .For this choice, the corresponding integral solutions of (1) are obtained as

$$x = x(a,b,p,q) = (2k+1)\left((2p+1+2q)(b^4 + 9a^4 - 18a^2b^2) + (2p - 6q - 3)(4ab^3 - 12ab)\right)$$

$$y = y(a,b,p,q) = (2k+1)\left((2q+1-2p)(b^4 + 9a^4 - 18a^2b^2) + (2p + 6q + 3)(4ab^3 - 12ab)\right)$$

$$z = z(a,b,p,q) = \left((2p)^2 + 3(2q+1)^2\right)(b^2 + 3a^2)$$

3. REMARKABLE OBSERVATIONS

I.: Let (x_0, y_0, z_0) be any given non- zero solution of (1).Then each of the following triples

$(x_{2n-1}, y_{2n-1}, z_{2n-1}) = (a^{4n-2}y_0, a^{4n-2}x_0, a^{2n-1}z_0)$, $(x_{2n}, y_{2n}, z_{2n}) = (a^{4n}x_0, a^{4n}y_0, a^{2n}z_0)$ also satisfy (1).

A few interesting relations observed from the above triples are presented below.

1. $$\frac{x_{2n-1}}{y_0} = \frac{y_{2n-1}}{x_0} = \left(\frac{z_{2n-1}}{z_0}\right)^2$$

2. $$\left(\frac{x_{2n-1}}{y_0}, \left(\frac{z_{2n-1}}{z_0}\right)^2, \frac{y_{2n-1}}{x_0}\right)$$ forms an Arithmetic Progression

3. $\left(\frac{x_{2n}}{x_0}, \left(\frac{z_{2n}^2}{z_0^2}\right), \frac{y_{2n}}{y_0}\right)$ forms an Arithmetic Progression

4. Each of the following is a nasty number.

(a) $6\left(\frac{x_{2n}}{x_0}\frac{y_{2n}}{y_0}\right)$

(b) $6\left(\frac{x_{2n-1}}{y_0}\frac{y_{2n}}{y_0}\right)$

(c) $6\left(\frac{x_{2n}}{x_0}\frac{y_0}{x_{2n-1}}\right)$

(d) $6\left(\frac{x_{2n-1}}{y_0}\frac{y_{2n-1}}{x_0}\right)$

5. Each of the following is a cubical integer.

(a) $a^2\left(\frac{y_{2n-1}}{x_0}\frac{z_{2n}}{z_0}\right)$

(b) $a^2\left(\frac{x_{2n-1}}{y_0}\frac{z_{2n}}{z_0}\right)$

6. Each of the following is a biquadratical integer.

(a) $a\left(\frac{z_{2n-1}}{z_0}\frac{z_{2n}}{z_0}\right)$

(b) $a^2\left(\frac{x_{2n-1}}{y_0}\frac{y_{2n}}{y_0}\right)$

7. In particular, when $a = 2^k$, $k > 1$, a few results observed are as follows:

(a) $\frac{y_{2n}}{y_0} = 3J_{4kn} + 1$

(b) $\frac{z_{2n}}{z_0} = j_{2kn} - 1$

(c) $\frac{z_{4n}}{z_0} + 2\frac{z_{2n}}{z_0} - 1 = ky_{2n}$

(d) $\left(\frac{2^{4k}-1}{2^{4k}}\right)\left(\sum_{n=1}^{N}\frac{x_{2n}}{x_0}\right) + 1 = (j_{2k}-1)\frac{x_{2n-1}}{y_0} = (3J_{2k}+1)\frac{y_{2n-1}}{x_0} = \frac{x_{2n}}{x_0}\frac{x_{2n-1}}{y_0}$

4. CONCLUSION

To conclude, one may search for other pattern of solutions and their corresponding properties.

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Gopalan et al.
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