DISCOVERY

Modified Inverse Lomax Distribution: Model and properties

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ABSTRACT

Here, a new distribution having three called *Modified Inverse Lomax Distribution* is proposed. Important statistical properties like the survival, hazard rate, the probability density function (PDF) is studied. Least Square, Cramer-Von Mises and Maximum Likelihood estimation methods are used for estimation of parameters using R programming software. A data set is discussed and performed the goodness-of-fit to assess the application of the proposed distribution. Various methods of model comparison and model validation are also used. The proposed model called *Modified Inverse Lomax Distribution* is more applicable as compared to some existing probability model.

Keywords: Lomax distribution, Estimation, Hazard function, Cramer-von Mises, maximum likelihood.

1. INTRODUCTION

Probability distribution helps to simulate the real-life problems and to compute the real-life data precisely and effectively. During last decade, different new probability distributions are introduced by researchers using different techniques. These techniques may me adding some extra parameters to distribution, merging the distribution or inverting the variables etc. These methods make new distribution more flexible and useful than the existing distributions. Many standard techniques to analyze the problem of real-life data are available; we still need models to solve the problem more effectively and precisely. That is in all the cases, classical techniques are not effective as the new distributions in literature, we can find numerous distributions. We can derive parametric distribution by changing the number of parameters to existing distribution such as Lomax and exponential family of distribution (Marshall and Olkin, 1997). Marshall and Olkin family of distribution Ghitany et al., (2007) is used to extend Lomax distribution.

We can find different new distributions derived by using the Lomax distribution. Power Lomax distribution is more flexible than existing Lomax distribution with decreasing and inverted bathtub hazard rate functions (Rady et al., 2016). Taking alpha as power, a new distribution named alpha power inverted exponential distribution was introduced using inverted exponential distribution (Ceren et al., 2018). Using Lomax random variable as a generator Ogunsanya et al., (2019), Odd Lomax exponential (type III) distribution was introduced. Compounding of odd generalize exponential using inverted Lomax distribution for obtaining Odd generalized exponentiated Inverse Lomax distribution (Maxwell et al., 2019). Lomax exponential distribution is obtained using Lomax distribution (Ijaz and Asim, 2019). The inverse Lomax- exponentiated G- family Falgore and Doguwa, (2020) is introduced using Inverse Lomax distribution. In real life, we can find many data that have bathtub-shaped hrf. In literature we can also find many modifications of Weibull distribution. Two parameter Weibull distributions is given as

$$\overline{F}(x,\alpha,\beta)=\exp[-(\alpha,x)]^{\beta}$$

This distribution is modified to generate several distributions that possess bathtub hrf. Modifications of Weibull distribution is exponentiated Weibull distribution (Mudhokar and Srivastav, 1993). As given by Lai et al., (2003) to get new lifetime distributions as

$$\overline{F}(x)=\exp[ax^{b}.\exp(\lambda x)]$$

We have modified the Modified Lomax distribution to introduce new probability model called, Modified Inverse Lomax distribution where inverse of the variable is taken. CDF and PDF of two parameters Lomax distribution (Pathak and Chaturvedi, 2013) is,

$$F(x)=1-\left(1+\beta x\right)^{-\lambda} \quad ;x>0,(\beta,\lambda)>0,$$

and, $$f(x,\beta,\lambda)=\lambda\beta\left(1+\beta x\right)^{-(\lambda+1)} \quad ;x>0,\beta>0,\lambda>0$$

An extra parameter α is added to modify the Lomax distribution as Modified Inverse Lomax distribution. CDF and PDF proposed model is given as.

$$F(x)=\left[1+(\beta/x)e^{-\lambda x}\right]^{-\alpha},\ \ x>0,\ \ \alpha,\beta,\lambda>0.$$

And the PDF of Modified Inverse Lomax Distribution is given as

$$f(x)=\alpha(\beta/x^{2}))(1+\lambda x)e^{-\lambda x}\left[1+(\beta/x)e^{-\lambda x}\right]^{-(\alpha+1)};\ \ x>0,\ \ \alpha,\beta,\lambda>0$$

The Modified Inverse Lomax (MILX) distribution

A three parameters Modified Inverse Lomax distribution with CDF and PDF are as follows

$$F(x)=\left[1+\left(\frac{\beta}{x}\right)e^{-\lambda x}\right]^{-\alpha},\ \ x>0,\ \ \alpha,\beta,\lambda>0. \tag{1}$$

And pdf of MILX is,

$$f(x)=\alpha\left(\frac{\beta}{x^{2}}\right)\left[1+(\beta x^{-1})e^{-\lambda x}\right]^{-\alpha-1}(1+\lambda x)e^{-\lambda x};\ \ x>0,\ \ \alpha,\beta,\lambda>0 \tag{2}$$

Survival function

Survival function of MILX is

$$S(x)=1-\left[1+(\beta/x)e^{-\lambda x}\right]^{-\alpha};\ x>0,\ \ \alpha,\beta,\lambda>0 \tag{3}$$

Hazard rate and reversed hazard rate function

MILX model has hrf as

$$h\left(x\right)=\frac{f\left(x\right)}{1-F\left(x\right)};\ 0<x<\infty$$

$$=\alpha(\beta/x^{2}))\left[1+(\beta/x)e^{-\lambda x}\right]^{-(\alpha+1)}(1+\lambda x)e^{-\lambda x}\left[1-\left[1+(\beta/x)e^{-\lambda x}\right]^{-\alpha}\right]^{-1} \tag{4}$$

Reverse hazard function of MILX is

$$h_{rev}(x)=\frac{f\left(x\right)}{F\left(x\right)}$$

$$=\alpha(\beta/x^{2}))(1+\lambda x)e^{-\lambda x}\left[1+(\beta/x)e^{-\lambda x}\right]^{-(\alpha+1)}\left[1+(\beta/x)e^{-\lambda x}\right]^{\alpha} \tag{5}$$

The various shapes of pdf and hazard rate function of MILX(α,β,λ) for at various values of parameters are displayed below in (Figure 1).

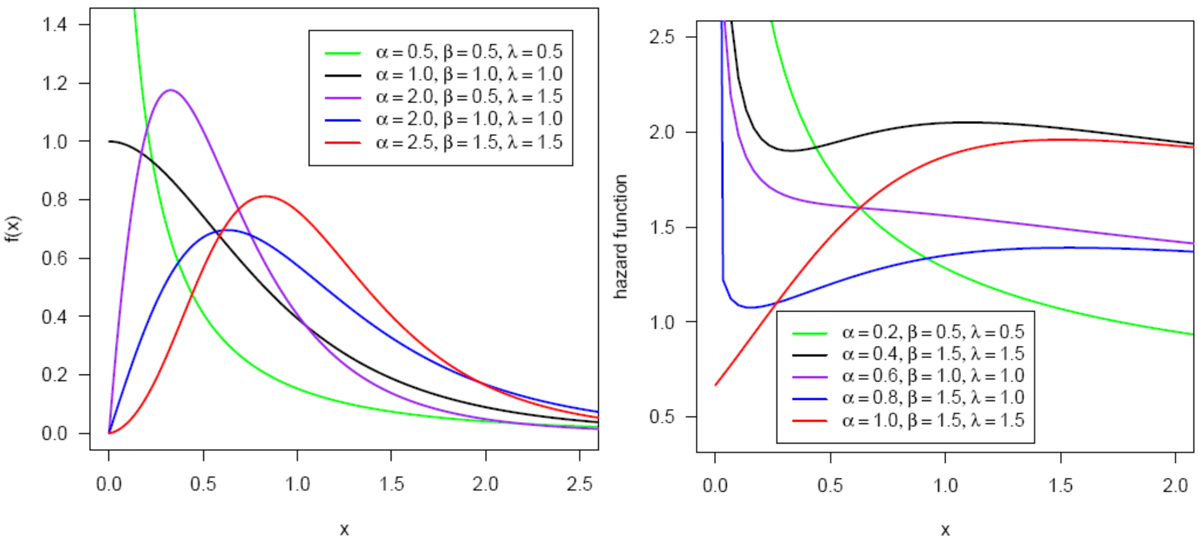


Figure 1 Graphs of PDF (left) and Hazard rate function (right) for some values of parameters

Here we have discussed the quantile function also. Quantile function can be defined here by expression (6) where X is non negative random variable.

$$Q_X(p) = F_X^{-1}(p)$$

$$\log\beta - \log x - \lambda x + 1 - p^{(-1/\alpha)} = 0 \tag{6}$$

Skewness and Kurtosis

Quartile based coefficient of skewness defined by Bowley's is,

$$S_B = \frac{\mathrm{Q}(3/4)\text{-}2\mathrm{Q}(1/2)+\mathrm{Q}(1/4)}{\mathrm{Q}(3/4)\text{-}\mathrm{Q}(1/4)}$$

Also, Moors, (1988), defined coefficient of kurtosis based on octiles can be defined as

$$\mathrm{K(Moors)} = \frac{\mathrm{Q}(0.875)\text{-}\mathrm{Q}(0.625)\text{-}\mathrm{Q}(0.125)+\mathrm{Q}(0.375)}{\mathrm{Q}(0.75)\text{-}\mathrm{Q}(0.25)}$$

2. PARAMETER ESTIMATION

Maximum Likelihood Estimation

Here, we have presented the ML estimators (MLE's) of the MILX model are estimated by using MLE method. Let a sample $\underline{x} = (x_1,\ldots,x_n)$ drawn randomly from MILX(α,β,λ) with log likelihood function,

$$\ell = n\log\alpha + n\log\beta + \sum_{i=1}^{n}\log(1+\lambda x_i) - 2\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i \\ -(\alpha+1)\sum_{i=1}^{n}\log\left[1+(\beta/x)_i\, e^{-\lambda x}\right] \tag{7}$$

Differentiating equation (7) with respect to parameters as

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(1+(\beta/x_i)e^{-\lambda x_i}\right)$$

$$\frac{\partial \ell}{\partial \beta}=\frac{n}{\alpha}-(\alpha+1)\sum_{i=1}^{n}\frac{e^{-\lambda x_i}}{x_i\left(1+\left(\beta/x_i\right)e^{-\lambda x_i}\right)}$$

$$\frac{\partial \ell}{\partial \lambda}=\beta(\alpha+1)+\sum_{i=1}^{n}\left(\frac{x_i}{1+\lambda x_i}\right)-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{\exp(-\lambda x_{(i)})}{\left[1+\left(\beta/x_i\right)e^{-\lambda x_i}\right]}$$

Equating $\frac{\partial \ell}{\partial \alpha}=\frac{\partial \ell}{\partial \beta}=\frac{\partial \ell}{\partial \lambda}=0$ and solving simultaneously for all parameters, ML estimators of the MILX$\left(\alpha,\beta,\lambda\right)$ distribution can be obtained. Normally, it is not possible of solving non-linear equations above so with the aid of suitable computer software one can solve them easily. Let $\underline{\Theta}=(\alpha,\beta,\lambda)$ denote vector of parameters of model MILX$\left(\alpha,\beta,\lambda\right)$ and the corresponding MLE of $\underline{\Theta}$ as $\underline{\Theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$ thus asymptotic normality becomes, $\left(\underline{\Theta}-\underline{\Theta}\right)\rightarrow N_3\left[0,\left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$. $I\left(\underline{\Theta}\right)$ is the Fisher's information matrix defined as,

$$I\left(\underline{\Theta}\right)=-\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \alpha . \partial \lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta . \partial \lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial \lambda . \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) \end{pmatrix}$$

Generally, $\underline{\Theta}$ is not known. That is, MLE has an asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$ is worthless. Asymptotic variance can be approximated by plugging in the estimated value of parameters. We have used $O(\underline{\Theta})$ as an estimate of $I\left(\underline{\Theta}\right)$ and is given by

$$O(\underline{\Theta})=-\begin{pmatrix} \frac{\partial^2 l}{\partial \hat{\alpha}^2} & \frac{\partial^2 l}{\partial \hat{\alpha}\partial \hat{\beta}} & \frac{\partial^2 l}{\partial \hat{\alpha}\partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\beta}\partial \hat{\alpha}} & \frac{\partial^2 l}{\partial \hat{\beta}^2} & \frac{\partial^2 l}{\partial \hat{\beta}\partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\lambda}\partial \hat{\alpha}} & \frac{\partial^2 l}{\partial \hat{\lambda}\partial \hat{\beta}} & \frac{\partial^2 l}{\partial \hat{\lambda}^2} \end{pmatrix}_{\left|(\hat{\alpha},\hat{\beta},\hat{\lambda},)\right.} = -H\left(\underline{\Theta}\right)_{\left|(\underline{\Theta}=\underline{\Theta})\right.}$$

Matrix H is called Hessian matrix.

Newton Raphson algorithm is used to optimize Likelihood function. Matrix below is the called Variance Covariance matrix,

$$\left[-H\left(\underline{\Theta}\right)_{\left|(\underline{\Theta}=\underline{\Theta})\right.}\right]^{-1}=\begin{bmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{cov}(\hat{\alpha},\hat{\lambda}) \\ \operatorname{cov}(\hat{\beta},\hat{\alpha}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\beta},\hat{\lambda}) \\ \operatorname{cov}(\hat{\lambda},\hat{\alpha}) & \operatorname{cov}(\hat{\lambda},\hat{\beta}) & \operatorname{var}(\hat{\lambda}) \end{bmatrix} \qquad (8)$$

Let $Z_{b/2}$ is upper percentile of SNV then using asymptotic normality of MLEs, approximated *100(1-b) %* CI of $\alpha,\beta,$ and λ of MILX$\left(\alpha,\beta,\lambda\right)$ is defined as,

$\hat{\alpha}\pm Z_{b/2}\sqrt{\operatorname{var}(\hat{\alpha})}$, $\hat{\beta}\pm Z_{b/2}\sqrt{\operatorname{var}(\hat{\beta})}$ and $\hat{\lambda}\pm Z_{b/2}\sqrt{\operatorname{var}(\hat{\lambda})}$.

Least Square Estimation

We have also used the LSE to estimate the α, β, & λ of MILX. For this we have to minimize function below

$$A(x\,|\,\alpha,\beta,\lambda)=\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2 \tag{9}$$

Let us consider that $F(X_{(i)})$ is CDF of order statistics $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ taking $\{X_1, X_2, \ldots, X_n\}$ as a sample having n items from a distribution function F (.). LSE of parameters are calculated by minimizing the function

$$A(x\,|\,\alpha,\beta,\lambda)=\sum_{i=1}^{n}\left[\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}-\frac{1}{n+1}\right]^2 \tag{10}$$

Differentiating (10), we get,

$$\frac{\partial A}{\partial\alpha}=2\sum_{i=1}^{n}\left[\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}-\frac{1}{n+1}\right]\log\left(1+\left(\frac{\beta}{x_{(i)}}\right)e^{-\lambda x_{(i)}}\right)^{-1}\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}$$

$$\frac{\partial A}{\partial\beta}=-2\alpha\sum_{i=1}^{n}\left[\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)\exp(-\lambda x_{(i)})\right)\right\}^{-\alpha}-\left(\frac{1}{n+1}\right)\right]\left(\frac{\exp(-\lambda x_{(i)})}{x_{(i)}}\right)\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)\exp(-\lambda x_{(i)})\right)\right\}^{-(\alpha+1)}$$

$$\frac{\partial A}{\partial\lambda}=2\alpha\beta\sum_{i=1}^{n}\exp(-\lambda x_{(i)})\left[\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)\exp(-\lambda x_{(i)})\right)\right\}^{-\alpha}-\frac{1}{n+1}\right]\left\{\left(1+\left(\frac{\beta}{x_{(i)}}\right)\exp(-\lambda x_{(i)})\right)\right\}^{-(\alpha+1)}$$

We can obtain weighted least square estimators by minimizing

$$B(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}W_i\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

Weights *Wi* are $W_i=\dfrac{1}{Var(X_{(i)})}=\dfrac{(n+1)^2(n+2)}{i(n-i+1)}$

Thus, weighted LSE of α, β, and λ are given by minimizing the function below,

$$B(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left\{\left(1+(\beta/x_{(i)})e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}-\frac{i}{n+1}\right]^2 \tag{11}$$

Cramer-Von Mises estimation

This method of estimation can be obtained by minimizing the function

$$Z(X;\alpha,\beta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left\{\left(1+(\beta/x_{(i)})e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left\{1-\left(1+\beta x_i e^{\alpha x}\right)^{-\lambda}\right\}-\frac{2i-1}{2n}\right]^2 \tag{12}$$

Differentiating (12), we get,

$$\frac{\partial Z}{\partial\alpha}=2\sum_{i=1}^{n}\left[\left\{\left(1+(\beta/x_{(i)})e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}-\frac{2i-1}{2n}\right]\log\left(1+(\beta/x_{(i)})e^{-\lambda x_{(i)}}\right)^{-1}\left\{\left(1+(\beta/x_{(i)})e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}$$

$$\frac{\partial Z}{\partial\beta}=-2\alpha\sum_{i=1}^{n}\left[\left\{\left(1+(\beta/x_{(i)})e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha}-\frac{2i-1}{2n}\right]\left(\frac{e^{-\lambda x_{(i)}}}{x_{(i)}}\right)\left\{\left(1+e^{-\lambda x_{(i)}}\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}^{-\alpha-1}$$

$$\frac{\partial Z}{\partial \lambda} = 2\alpha\beta\sum_{i=1}^{n} e^{-\lambda x_{(i)}}\left[\left\{\left(1+\left(\beta / x_{(i)}\right)e^{-\lambda x_{(i)}}\right)\right\}^{-\alpha} - \frac{2i-1}{2n}\right]\left\{\left(1+e^{-\lambda x_{(i)}}\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}^{-\alpha-1}$$

After solving non-linear equations $\frac{\partial Z}{\partial \alpha} = \frac{\partial Z}{\partial \beta} = \frac{\partial Z}{\partial \lambda} = 0$ CVM estimators can be obtained.

3. APPLICATION TO REAL DATASET

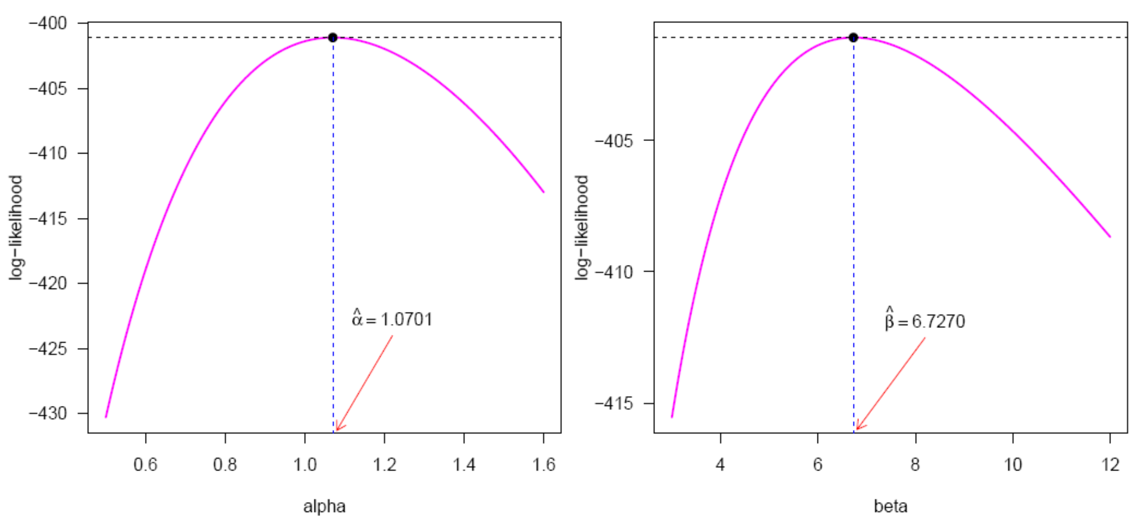
For applicability of the model, we have considered a real set data. Data is by Lee and Wang, (2003) (Statistical methods for survival data analysis, 3rd edn. Wiley, New York). Ramos et al., (2015) (The Kumaraswamy-G Poisson family of distributions. Journal of Statistical Theory and Applications, 14(3), 222-239).

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.2, 2.23, 0.26, 0.31, 0.73, 0.52, 4.98, 6.97, 9.02, 13.29, 0.4, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07, 0.22, 13.8, 25.74, 0.5, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36, 0.82, 0.51, 2.54, 3.7, 5.17, .28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65, 0.39, 10.34, 14.83, 34.26, 0.9, 2.69, 4.18, 5.34, 7.59, 10.66, 4.5, 20.28, 12.63, 0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83,4.33, 8.37, 3.36, 5.49, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76, 0.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.4, 5.85, 2.02, 12.07

By use of R software of the optim () function Team, (2020), calculation of MLEs of MILX by maximizing the log likelihood function defined in equation (5.3.1) (Mailund, 2017). Values determined are tabulated in (Table 1). Figure 2 displays the graph of profile log-likelihood function of parameters showing that ML estimates is uniquely defined.

Table 1 Estimated parameters using MLE, LSE and CVME

Method	alpha	beta	lambda
MLE	1.0701	6.7270	0.0589
LSE	0.8818	10.8745	0.0882
CVME	0.9001	10.6555	0.0899



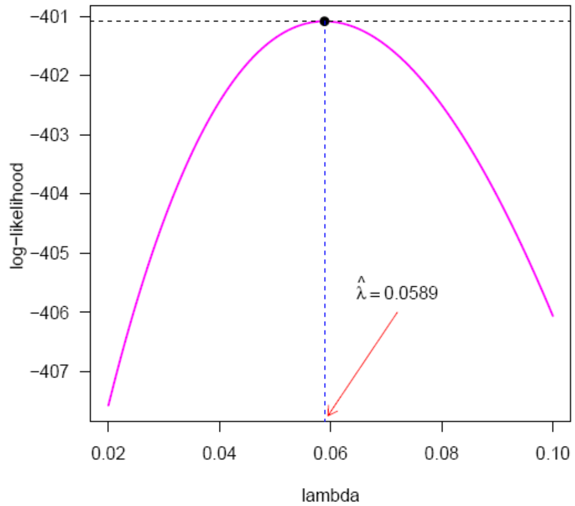


Figure 2 Profile Log-likelihood function of parameters.

Graph of P-P plot and Q-Q plots are shown in (Figure 3). It is found that MILX model has better fitting to the data taken in consideration (Figure 4). MLE, LSE and CVE methods are used for estimation and the estimated value of the parameters of MILX model with log-likelihood, AIC, BIC, CAIC, and HQIC criteria in (Table 2). KS, W and A2 statistic and calculated p-value are given in (Table 3).

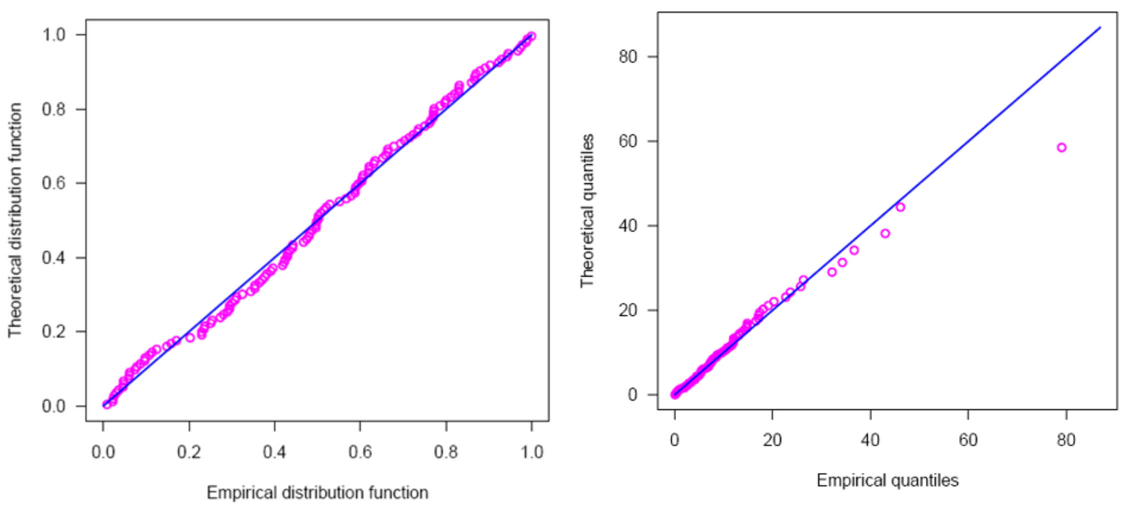


Figure 3 The P-P plot (left) & Q-Q plot (right) of the MILX Model.

Table 2 Estimated parameters with LL and values of information criteria

Method	alpha	beta	lambda	LL	AIC	BIC	CAIC	HQIC
MLE	1.0701	6.7270	0.0589	-401.084	808.1674	816.7235	808.3609	811.6438
LSE	0.8818	10.8745	0.0882	-402.620	811.2404	819.7965	811.4340	814.7168
CVME	0.9001	10.6555	0.0899	-402.759	811.5186	820.0746	811.7121	814.9949

Table 3 The KS, A2 and W statistic and their p-values

Method	KS (p -Values)	A2 (p -Values)	W (p -Values)
MLE	0.0433 (0.9702)	0.3678(0.8797)	0.0535(0.8556)
LSE	0.0347(0.9978)	0.3538(0.8928)	0.0297(0.9776)
CVE	0.0346(0.9979)	0.3774(0.8706)	0.0291(0.9793)

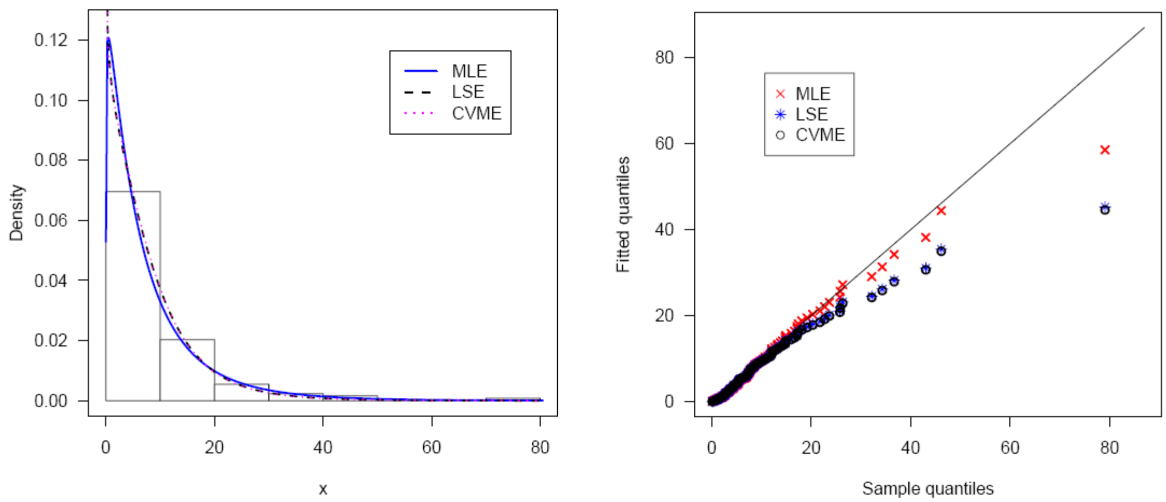


Figure 4 Histogram and density function of fitted distributions (left) & fitted quantile vs sample quantile (right) of different estimation methods of MILX.

In this part of study, applicability of MILX taking a real dataset used previously is presented. Model's potentiality is compared with following four distributions.

I. Generalized Exponential (GE) distribution**:**

The Pdf of GE model (Gupta and Kundu, 2007) is

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1}$$

II. Exponentiated Exponential Poisson distribution (EEP):

Pdf of EEP is, (Ristić and Nadarajah, 2014)

$$f(x) = \frac{\alpha\beta\lambda}{(1-e^{-\lambda})} e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}\exp\left\{-\lambda(1-e^{-\beta x})^{\alpha}\right\}$$

III. Generalized Exponential Extension (GEE) distribution**:**

Pdf of GEE is (Lemonte, 2013)

$$f_{GEE} = \alpha\,\beta\lambda\,(1+\lambda x)^{\alpha-1}\left[1-\exp\left\{1-(1+\lambda\,x)^{\alpha}\right\}\right]^{\beta-1}\exp\left\{1-(1+\lambda\,x)^{\alpha}\right\}$$

IV. Lindley-Exponential (LE) distribution:

The PDF of LE (Bhati et al., 2015) distribution is,

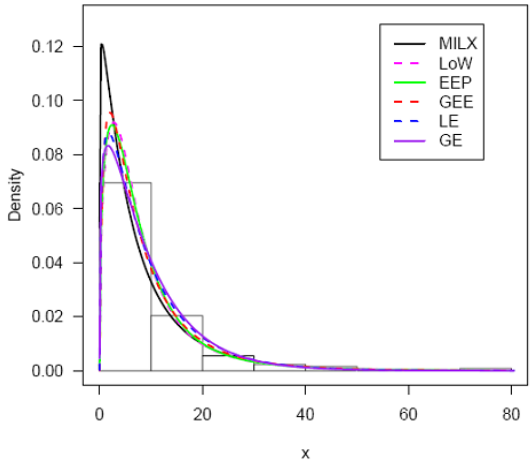
$$f_{LE}(x) = \lambda\left(\frac{\theta^2}{1+\theta}\right)\left\{1-\log\left(1-exp(-\lambda x)\right)\right\}exp(-\lambda x)\left(1-exp(-\lambda x)\right)^{\theta-1};$$

Various information criteria values for the testing of the applicability of the MILX are tabulated below (Table 4). We have displayed the graph of goodness-of-fit of MILX and models defined as (Figure 5), We have plotted the empirical distribution function of the proposed model MILX and the fitted distribution function and is shown in (Figure 6)

Table 4 AIC, CAIC, BIC, HQIC and log likelihood Values

Dist.	AIC.	CAIC	BIC	HQIC	ll
MILX	808.1674	808.3609	816.7235	811.6438	-401.0837
LoW	824.9487	825.1423	833.5048	828.4251	-409.4744
EEP	825.5056	825.6991	834.0617	828.9819	-409.7528

GEE	827.2026	827.3961	835.7586	830.6789	-410.6013
LE	828.0985	828.1945	833.8026	830.4161	-412.0493
GE	830.1552	830.2512	835.8592	832.4728	-413.0776



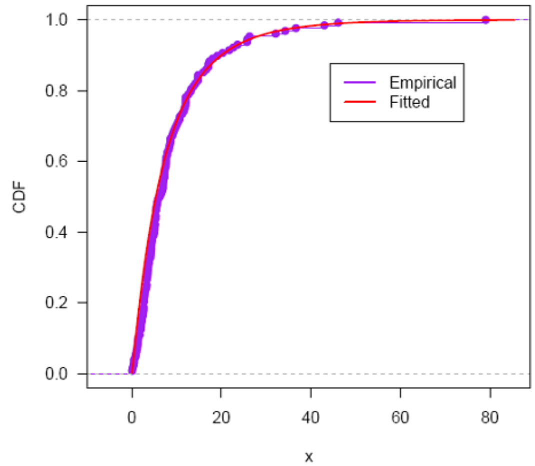


Figure 5 Histogram & pdf (left) of fitted model and Empirical versus estimated distribution function (right).

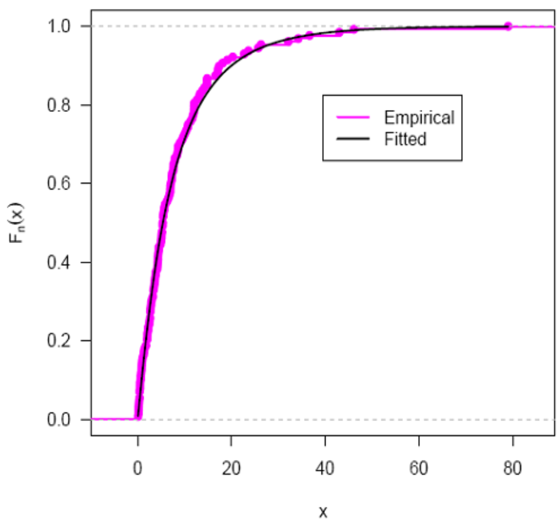


Figure 6 Empirical distribution functions versus the fitted distribution function of the MILX

Goodness-of-fit of the MILX model with other competing model are compared in this section. We have also displayed value of Kolmogorov-Simnorov test, Cramer-Von Mises and, Anderson-Darling test statistic in (Table 5). MILX has the minimum test statistic values. That is, p-values of MILX model are higher indicating the better and consistent fit than considered models.

Table 5 Test statistics and p-values

Dist.	K-S (p - Values)	W (p - Values)	A2 (p - Values)
MILX	0.0433(0.9702)	0.0535(0.8556)	0.3678(0.8797)
LoW	0.0321(0.9994)	0.0149(0.9997)	0.1010(0.9999)
EEP	0.0380(0.9925)	0.0220(0.9946)	0.1486(0.9987)
GEE	0.0441(0.9640)	0.0393(0.9370)	0.2631(0.9630)
LE	0.0622(0.7059)	0.0899(0.6377)	0.5250(0.7211)
GE	0.0725(0.5115)	0.1279(0.4652)	0.7137(0.5472)

4. CONCLUSION

Presented article explains a distribution named Modified Inverse Lomax distribution having three parameters. Detailed analysis of different statistical characteristics of model is also explained. We have presented expressions for its hazard rate function, survival function, the quantile function and skewness & kurtosis. Maximum likelihood estimation, Cramer-Von Mises estimation, and Least Square estimation methods are used to estimate the parameter. It is found that MLEs are better w.r.t. LSE and CVM. PDF curve of model has shown that it can have various shapes like increasing as well as decreasing. Monotonically increasing, constant as well as bathtub shaped based hazard function is seen. Applicability and suitability of model is evaluated. For this, we have considered a real-life dataset. We found distribution that MILX is much flexible.

Authors' Contributions

Lal Babu Sah Telee: Parameter Estimation, Model Comparison, Data analysis

Ram Suresh Yadav: Model Formulation, Introduction, Studying the properties and data analysis, References

Vijay Kumar: Model formulation, Graphics, Data analysis, Data analysis using R

Informed consent

Not applicable.

Ethical approval

Not applicable.

Conflicts of interests

The authors declare that there are no conflicts of interests.

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Data and materials availability

All data associated with this study are present in the paper.

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