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On the two different formulations of mechanical energy in driven oscillations

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ABSTRACT

We introduce two different formulae of mechanical energy of harmonic oscillators driven by an external force in this work. In the first formula, the energy is composed of two terms, i.e., the kinetic energy and quadratic-potential-energy terms. Another formula is that it is just the same as the Hamiltonian itself. While the first kind of energy reflects the mechanical energy remained in the system at a certain instant of time, the second energy depends on the linear potential term as well as the terms quadratic in canonical variables. Time evolution of the two energies is analyzed for two specific cases of which driving force is different from each other. The second kind of energy can be negative, whereas the first kind is always positive. The second mechanical energy has generally been used in optimal control theory in relation with shortcuts to adiabaticity.

Keywords: Mechanical energy, Harmonic oscillator, Hamiltonian, External force, Classical solutions

1. INTRODUCTION

Mechanical energy is a basic element in describing various dynamical systems including oscillatory ones. Transfer of energy for a system interacting with other ones or its environment has attracted much interest in the physics community for a long time (Mukherjee and Barbatti, 2022; Chen and Muga, 2010; Choi, 2021; Hioe et al., 1978; Koochi and Goharimanesh, 2021). Mechanical energy is evaluated from the Hamiltonian of the considered system in most cases as we know. However, mechanical energy is not always the same as the Hamiltonian for complicated systems (Kobe, 1990; Kobe et al., 1990). For instance, in damped oscillatory systems, mechanical energy and the Hamiltonian are different from each other by a time-dependent exponential factor (Colegrave et al., 1989; Yeon et al., 1987; Choi, 2013). The role of the Hamiltonian in such a case is confined to the generator of the equations of motion of the system. The relation between energy and the Hamiltonian is delicate, while such a theme is not actively discussed previously.

A notable consequence in this direction is that mechanical energy in driven oscillatory systems is defined in two ways in the literature. We will manage such two formulations of energy regarding their fundamental mathematical and physical aspects. Time evolution of the two energies will be analyzed for the cases where the type of the force is specifically chosen. From this, the differences and similarities of the two energies will be clarified. We invoke readers the trend of such duplicate energy formulations on one hand, since most of researchers do not recognize it yet.

2. MATERIALS AND METHODS

We will look at the methodology of defining two different mechanical energies for oscillatory systems in this section. To analyze the physical characteristics of the two energies, it is not necessary to introduce a complicated system. We consider a forced oscillator of which classical equation of motion is simply given by

$$\ddot{x} + \omega^2 x = f(t), \tag{1}$$

where ω is the angular frequency and $f(t)$ is an arbitrary force. The Hamiltonian that gives this equation of motion can be written as

$$H = \frac{p^2}{2m} + \frac{1}{2} m \, [\omega^2 x^2 - 2f(t)x] + g(t), \tag{2}$$

where $g(t)$ is a time function which is real. The mechanics relevant to this Hamiltonian, including its solutions, is well known. The classical solution of position (momentum) is composed of a complementary function x_c (p_c) and a particular solution x_p (p_p), such that

$$x_{cl} = x_c + x_p, \tag{3}$$
$$p_{cl} = p_c + p_p. \tag{4}$$

The complementary functions for position and momentum are independent of the force and are given by

$$x_c = x_0 \sin(\omega t + \theta), \tag{5}$$
$$p_c = m\,\omega x_0 \cos(\omega t + \theta), \tag{6}$$

where x_0 is an amplitude and θ is an arbitrary phase. However, the particular solutions are different depending on the form of $f(t)$.

Among the two classes of mechanical energy appeared in the literature for the above system, we first manage the one defined as

$$E = \frac{p_{cl}^2}{2m} + \frac{1}{2} m \, \omega^2 x_{cl}^2, \tag{7}$$

where x_{cl} and p_{cl} are given in equations (3) and (4), respectively. This type of energy was claimed by Landau and Lifshits (1981), Dybiec and Gudowska-Nowak (2017), Romero-Bastida and López (2020), Yeon et al. (2001), Um and Yeon (2002), etc. Yeon et al. (2001) and Um and Yeon (2002) treated mechanical energy of the forced damped harmonic oscillator, but, if the damping factor is removed from their treatment, their Hamiltonian and energy are reduced to equation (2) and equation (7) respectively. In particular, Romero-Bastida and López (2020) used this concept of energy in analyzing the performance of an energy harvester driven by colored noise, which converts the power supplied by external noise into electrical energy.

Another definition of mechanical energy is of the form

$$e = \frac{p_{cl}^2}{2m} + \frac{1}{2} m \, [\omega^2 x_{cl}^2 - 2f(t)x_{cl}] + g(t), \tag{8}$$

which is naively the same as the Hamiltonian. This kind of energy for the same system was adopted by Kuzmin and Robnik (2007), Guéry-Odelin et al. (2019), Guéry-Odelin and Muga (2014), Ding et al. (2020), Zhang et al. (2021), etc. This energy has especially been applied to shortcuts to adiabaticity, whose main concern is to find fast routes of final results in an adiabatic change by optimally controlling system parameters (Guéry-Odelin et al., 2019).

Let us call equations (7) and (8) as just the first energy and the second energy, respectively, for convenience from now on. The first energy is independent of $g(t)$ and the associated potential energy is measured from $x = 0$. However, the second energy is different depending on $g(t)$, and the reference point of the associated potential energy is different from that position. To see this in more detail, let us express equation (8) as

$$e = \frac{p_{\mathrm{cl}}^2}{2m} + \frac{1}{2} m\,\omega^2 [x_{\mathrm{cl}} - a(t)]^2 - b(t), \qquad (9)$$

where $a(t) = f(t)/\omega^2$ and $b(t) = mf^2(t)/(2\omega^2) - g(t)$. We can think from this that potential energy in the second definition is measured from the reference position $x = a(t)$ and $-b(t)$ is added in the energy as a compensation. $a(t)$ is used as a control function in shortcuts to adiabaticity, whereas $b(t)$ acts as a gauge function (see, for example, p. 37 of Ref. (Guéry-Odelin et al., 2019)).

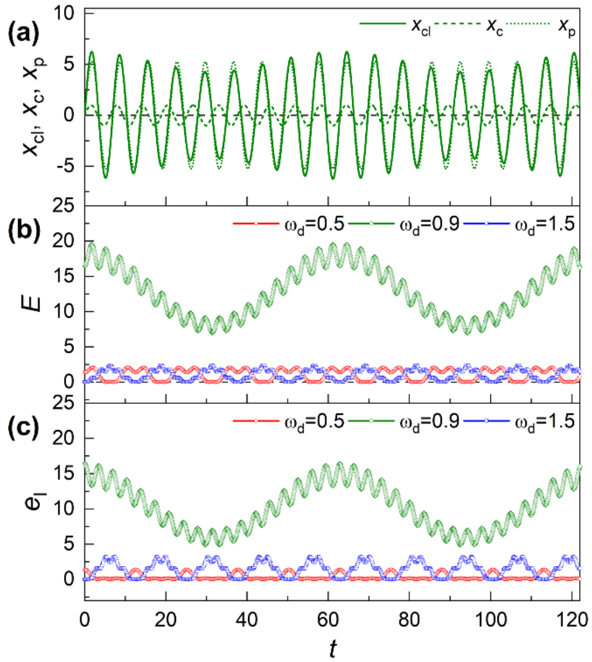

Figure 1 Mechanical energies with the choice of $f(t)$ as equation (10) and $g(t) = 0$. (a): time evolution of x_{cl} with x_{c} and x_{p}, where we have chosen $\omega_{\mathrm{d}} = 0.9$. (b, c): time evolution of energy where (b) is the one defined in equation (7) and (c) is the second energy e_{I}. Panels (b) and (c) were drawn for several different values of ω_{d} as can be seen from the legend. The parameters that we used are $\omega = 1$, $x_0 = 1$, $f_0 = 1$, $m = 1$, and $\theta = \phi = 0$: These values are chosen to be dimensionless for convenience and this convention will also be used in the subsequent figures.

3. RESULTS AND DISCUSSION

Analyses with the choice of $g(t) = 0$

We now see the two energies with the choice of $g(t) = 0$. Kuzmin and Robnik (2007) investigated how energy evolves when the oscillatory system is nonconservative with this choice of $g(t)$. Since the second energy is different depending on $g(t)$, let us denote it as e_{I} instead of e in this case. We consider two types of $f(t)$. At first, we put the force as a sinusoidal form such that

$$f(t) = f_0 \sin(\omega_{\mathrm{d}} t + \phi), \qquad (10)$$

where f_0 is the strength of the force and ω_{d} is a driving angular frequency. While complementary functions are common regardless of the type of the chosen force, particular solutions are different depending on $f(t)$ as mentioned earlier. The particular solutions associated with the force of equation (10) are given by

$$x_{\mathrm{p}} = \frac{f_0}{\omega^2 - \omega_{\mathrm{d}}^2} \sin(\omega_{\mathrm{d}} t + \phi), \qquad (11)$$

$$p_{\mathrm{p}} = \frac{m\,\omega_{\mathrm{d}} f_0}{\omega^2 - \omega_{\mathrm{d}}^2}\cos(\omega_{\mathrm{d}} t + \phi). \quad (12)$$

From Figure 1, we see that the evolutions of the two energies are somewhat different from each other. For the case of $\omega_{\mathrm{d}} = 0.9$, the scales of energies are very large compared to other two cases where $\omega_{\mathrm{d}} = 0.5$ and $\omega_{\mathrm{d}} = 1.5$. This outcome is due on account of the near resonance of the driving frequency ω_{d} with the natural frequency ω. The scale of E is larger than e_{I} in this case. Both energies fluctuate fairly regularly according to the variation of the oscillatory amplitude caused by the interference of the system with the external driving field.

In particular, for the case of $\omega_{\mathrm{d}} = 0.9$, the oscillations of energies are two types: the main oscillation with large amplitude (but with a small frequency) and the sub oscillation with small amplitude (but with a large frequency). The main oscillation accompanies the beat of x_{cl}, which has taken place by the relative smallness of the difference between ω and ω_{d}. The sub oscillation has taken place on account of the direction of the exerting external force. That is, energy increases when the direction of the force is the same as the moving direction of the oscillator, whereas energy decreases when the direction of the force is opposite to the direction that the oscillator moves. Because such two relative directions occur alternately, the energies fluctuate with the sub-oscillatory angular frequency.

The energies with $\omega_{\mathrm{d}} = 0.5$ and $\omega_{\mathrm{d}} = 1.5$ also evolve periodically but with complicated fluctuations. Such fluctuations also occurred depending on whether the direction of the force is the same or different compared to that of the oscillator motion. We can clearly confirm the difference in the evolution patterns of E and e_{I} for these two cases of ω_{d} from red and blue curves in Figure 1(b, c), which are non-resonant with ω.

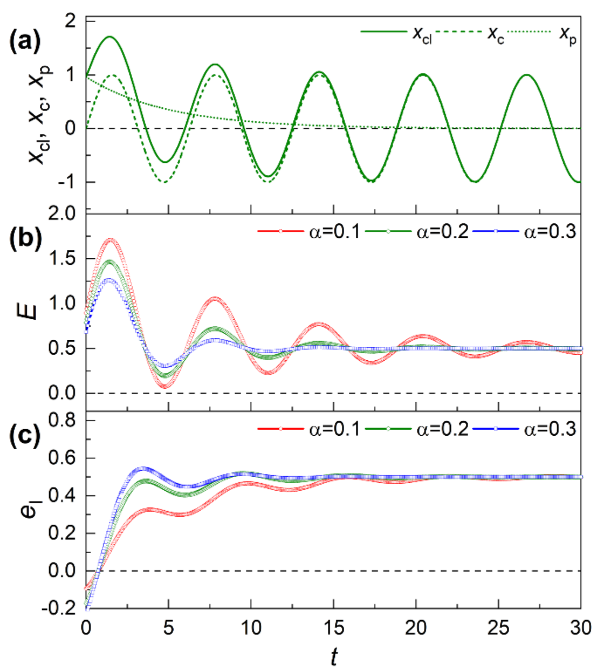

Figure 2 Mechanical energies with the choice of $f(t)$ as equation (13) and $g(t) = 0$. (a): time evolution of x_{cl} with x_{c} and x_{p}, where we have chosen $\alpha = 0.2$. (b, c): time evolution of the two energies (b) E and (c) e_{I}. Panels (b) and (c) were drawn for several different values of α. The chosen scales of parameters are $\omega = 1$, $x_0 = 1$, $f_0 = 1$, $m = 1$, and $\theta = 0$.

We now take the force, as another example, in the form

$$f(t) = f_0 e^{-\alpha t}, \quad (13)$$

where α is a real constant. The particular solutions in this case are given by

$$x_{\mathrm{p}} = \frac{f_0}{\omega^2 + \alpha^2} e^{-\alpha t}, \tag{14}$$

$$p_{\mathrm{p}} = -\frac{m\,\alpha f_0}{\omega^2 + \alpha^2} e^{-\alpha t}, \tag{15}$$

whereas the complementary functions are the same as equations (5) and (6). We see from Figure 2 that the evolution of the two energies is different from each other in this case, especially at some initial instant of time. E oscillates due to the effect of the force, where amplitude of such an oscillation decreases with time. This means that the oscillator gives and takes energy with the external field that induces the force. Such an exchange of energy is gradually attenuated by the decrease of the force as time goes by. Oscillation of mechanical energy also appears when the oscillatory system is driven by a frictional force instead of the external force (Choi, 2021). Any force exerted on a system may induce a change of energy in the system in general.

On the other hand, e_{I} is slightly lower than zero around $t = 0$, but it rapidly grows with the progress of time until it reaches a quasi-stable value that fluctuates a little. e_{I} no longer varies soon after that. In summary, the fluctuations of both energies gradually disappear and reach a constant value as the force disappears exponentially. It is noticeable that e_{I} can be a negative value as can be seen from Figure 2(c), whereas E is always positive.

For the case of $g(t) = mf^2(t)/(2\omega^2)$

Let us see the case that $g(t) = mf^2(t)/(2\omega^2)$, which corresponds to $b(t) = 0$. The second mechanical energy in this case is reduced to

$$e_{\mathrm{II}} = \frac{p_{\mathrm{cl}}^2}{2m} + \frac{1}{2} m\,\omega^2 [x_{\mathrm{cl}} - a(t)]^2. \tag{16}$$

The subscript II in the symbol of this energy is inserted in order to distinguish it with the energy in the previous subsection. We can confirm that this energy is always positive, whereas the energy e_{I} can be negative as we have seen previously. The energy in equation (16) was used by some authors in order to formulate a minimal-time transportation theory for a particle confined in a harmonic trap (Guéry-Odelin and Muga, 2014; Ding et al., 2020; Zhang et al., 2021). The time evolution of e_{II} is given in Figure 3 for both cases of $f(t)$ represented in equations (10) and (13). From this figure, we see that e_{II} is larger than e_{I}, but smaller than E in most instants of time.

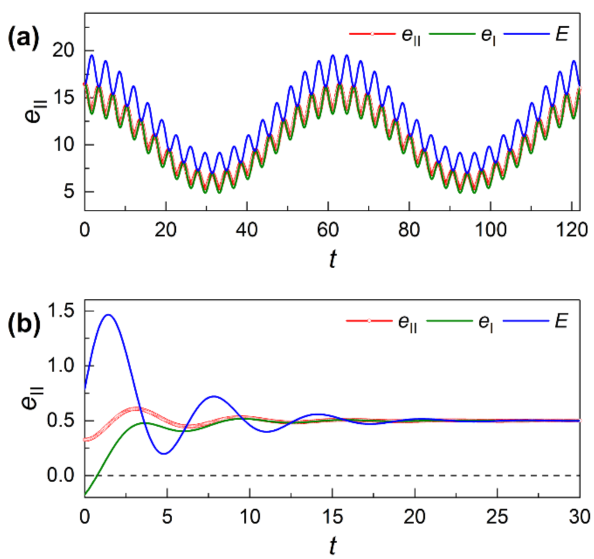

Figure 3 Time evolution of e_{II} given in equation (16) and its comparison with e_{I} and E for the case where $f(t)$ is chosen as equation (10) for (a) and as equation (13) for (b). Panel (a) was drawn with the choice of $\omega_{\mathrm{d}} = 0.9$, $\omega = 1$, $x_0 = 1$, $f_0 = 1$, $m = 1$, and $\theta = \phi = 0$. (b) was drawn with the choice of $\alpha = 0.2$, $\omega = 1$, $x_0 = 1$, $f_0 = 1$, $m = 1$, and $\theta = 0$.

4. CONCLUSIONS

We analysed two different formulae of mechanical energy for forced oscillatory systems. The time evolutions of the two energies were compared with each other under specific choices of the external force. This comparison was carried out for the first time as far as we know. The first energy is basic where it is composed of the kinetic and the potential energies that are given in terms of p^2 and x^2, respectively. Because square of canonical variables is positive, this energy is always positive.

The second mechanical energy for a forced simple harmonic oscillator is a sort of mimic the Hamiltonian. Because the energy associated with the driving force can be negative, e_{I} treated in Sec. 3 can be a negative value. On the other hand, e_{II} managed in the same section is always positive. Although we have used a basic system in our analysis, this research may make researchers rethink the description of energy and its relation with the external force from a fundamental physical level.

Authors contributions

Both authors contributed equally.

Ethical approval

Not applicable.

Informed consent

Not applicable.

Conflicts of interests

The authors declare that there are no conflicts of interests.

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Data and materials availability

All data associated with this study are present in the paper.

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