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The economic design of the VSS T^2 control scheme

Kazemzadeh RB[1], Faraz A[2]

1.Faculty of Industrial Engineering, Tarbiat Modares University, Tehran, Iran: Email:- rkazem@modares.ac.ir
2.Faculty of Industrial Engineering, Tarbiat Modares University, Tehran, Iran

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ABSTRACT

In this paper, the VSS T^2 control chart for monitoring the process mean is economically designed. The cost model proposed by Costa and Rahim (2001) to economically design the variable parameter $\overline{X}$ control chart is used here and it is minimized through a genetic algorithm (GA) approach. Then a comparison between the FRS and VSS schemes are made to clarify the economic advantages of the VSS scheme.

Keywords: Hotelling's T^2 control chart, Variable Sample size (VSS) scheme, Economic Design, Markov chain approach, Genetic Algorithm (GA).

1. INTRODUCTION

Nowadays there is growing interest in multivariate statistical quality control, i.e. the simultaneous control of several related quality characteristics of a process, because quality control problems in industry may involve more than a single quality characteristic, i.e. a vector of characteristics. This has formed the basis of extensive work performed in the field of multivariate quality control. Shewhart, who is famous for the development of the statistical control chart (Shewhart charts) first recognized the need to consider quality control problems as multivariate in character. A great deal of work on multivariate statistical control procedures was performed in the 1930's and in the 1940's by Hotelling (1947). Jackson (1985) mentioned in his paper that the multivariate techniques should posse three important properties: (1) they produce a single answer to the question: is the process is in-control? , (2) has the specified type I error been mentioned?, and (3) these techniques must take into account the relationship between the variables. The Hotelling's T^2 chart, satisfies the above three properties and also has the advantage of its simplicity. Consider p correlated characteristics are being measured simultaneously and are being controlled jointly. It is assumed that the joint probability distribution of the p quality characteristics is the p-variate normal distribution. The procedure requires computing the samples mean for each of the *p* quality characteristics from a sample of size n_0. The subgroup statistics $T^2 = \mathbf{n_0}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' \Sigma^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$ are plotted on a control chart in sequential order. Here, $\boldsymbol{\mu}_0$ is the vector of in-control means for the p quality characteristics. For the sake of simplicity, in this paper, we assume here that the process mean vector $\boldsymbol{\mu}_0$ and variance-covariance matrix Σ are known. The statistic T^2 is distributed as a chi-square variable with *p* degrees of freedom. The upper limit on the control chart is $UCL = x_{\sim}^{2}(p)$. When the process is in control, with a probability ~ , the statistic T^2 exceeds the UCL, so that the overall error rate (type I) can be maintained exactly at the level ~ . If the process is out of control, the chart statistic is distributed as a non-central chi-square distribution with *p* degrees of freedom with non-centrality parameter $a = n_0 d^2$, where *d* is the Mahalanobis distance that is used as a measure of process shift in multivariate statistical quality control. It is usually assumed that the variance-covariance structure of the quality characteristics being charted does not change and that the assignable cause is manifested by a shift or drift in at least one component of the mean vector of the process.

The magnitude of this shift is often expressed by $d^2 = (\boldsymbol{\mu} - \boldsymbol{\mu}_0)' \Sigma^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)$, and $\boldsymbol{\mu}$ is the *p*-characteristics mean vector 7. The traditional practice in applying a control chart to monitor a process is to obtain samples of fixed size, n_0. Therefore, traditional control charts use a Fixed Ratio Sampling (FRS) scheme. Variable Sample Size procedure (VSS) is a scheme that varies the sampling size as a function of prior sample results. The design of univariate Shewhart charts with adaptive sample sizes studied by Burr (1969), Daudin (1992), Prabhu et al. (1993), and Costa (1994). In these studies, two sample sizes are used and this procedure shows better power in detecting shifts in the mean. Zimmer et al. (1998) present a three-state adaptive sample size control chart and compared it with both the standard Shewhart control chart and the developed two-state adaptive sample size control chart. They conclude that three-state procedure is only slightly better than the two-state scheme, and so the two-state scheme is likely adequate in most applications 7. In the Multivariate SPC, The FRS T^2 control chart shows a good performance to detect large shifts in the process mean. However, in many practical situations it is necessary to be able to detect even moderate shifts in the process mean. In such cases, the statistical efficiency of the FRS T^2 chart (in terms of the speed with which process mean shifts are detected) is poor. Aparisi (1996) studied the VSS T^2 control chart. He considered an adaptive strategy for the subgroup size based on the data trends. He divided the area between the UCL and the origin, into two zones for the use of two different sample sizes. If the current sample value falls in a particular zone, then the corresponding sample size is to be applied for the successive sampling. He showed that the Hotelling's T^2 control chart with VSS scheme, significantly improves the efficiency of the standard Hotelling's T^2 control chart in detecting small changes in the process mean. For moderate shifts the obtained Average Run Length (ARL) values are about 1/3 of FRS scheme. However, Aparisi (1996) found that, when the shifts are large, ARL for the FRS chart is smaller or almost equal for the VSS chart 7.

Aparisi (1996) assumed that the process goes out of control in the beginning of the monitoring the process. Faraz and Moghadam (2008) extended the work of Aparisi (1996) to the case in which the shift in the process mean does not occur at the beginning but at some random time in the future. Further, they assumed that the occurrence time of the shift is an exponentially distributed random variable. They showed that the proposed modification can improve the power of the chart to detect small to specifically moderate shifts in the process mean. Recently the economic design (ED) of control charts is of a great importance. Based on the ED procedure, the charts is designed in such a way that the overall costs associated with maintaining current control of a process is minimized. Weiler (1952) was one of the first researchers who attempted to design a control chart based on an economic viewpoint. He suggested that the optimal sample size should be the amount that minimizes the total number of inspected items required to detect a shift in the process mean. Four years later, Duncan (1956) proposed a very popular procedure for the ED of Shewhart $\bar{X}$ chart. Lorenzen and Vance (1986) developed an economic model for all kinds of control charts, not just the $\bar{X}$ chart. These economic models consider classical sampling scheme (FRS) and adopting these models to variable ratio sampling (VRS) procedures seems somewhat difficult. Costa and Rahim (2001) developed an economic model based on the Markov chain approach which is suitable to study the economic design of the VRS schemes.

Montgomery (1980) made a very good literature review on ED of control charts and listed fifty one references on the topic. Woodall (1986) criticized the EDs by noting that these models result in designs with high Type I error. Saniga (1989) amended this drawback by adding some statistical constraints on the Type I and/or Type II errors. He called this approach the Economic Statistical Design (ESD). He observed that the ESD statistical properties are as good as statistically designed control charts. We hereby propose the economic design of the VSS T^2 control schemes which has not been found in the literature. The paper is organized as follows: In section 2 the VSS T^2 control scheme proposed by Faraz and Moghadam (2008) is briefly reviewed and upon the Markov chain approach, the cost model proposed by Costa and Rahim (2001) is constructed. Section 3 makes a comparison between the FRS and VSS schemes. Finally, concluding remarks make up the last section.

2. THE COST MODEL

Consider *p* correlated quality characteristics are to be monitored simultaneously using the T^2 scheme. First, it is assumed that the joint probability distribution of the quality characteristics is a *p*-variate normal distribution with the known in-control mean vector $\vec{\boldsymbol{\mu}}_0$ and known variance-covariance matrix Σ. Then the subgroups' statistic $T_i^2 = \mathbf{n}(\bar{\vec{\mathbf{x}}}_i - \vec{\boldsymbol{\mu}}_0)' \Sigma^{-1} (\bar{\vec{\mathbf{x}}}_i - \vec{\boldsymbol{\mu}}_0)$ (each of size *n*) is plotted in sequential order and the chart signals as soon as $T_i^2 \geq k = \chi^2_{\alpha}(p)$. To gain from the Markov chain properties, following transient states are defined:

State 1: $0 \leq T^2 < w$ and the process is in control;

State 2: $w \leq T^2 < k$ and the process is in control;

State 3: $T^2 \geq k$ and the process is in control;

State 4: $0 \leq T^2 < w$ and the process is out of control;

State 5: $w \leq T^2 < k$ and the process is out of control;

State 6: $T^2 \geq k$ and the process is out of control.

The state 3 stands to count the false alarms and the state 5 represents the absorbing state. Hereafter, the transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where p_{ij} denotes the transition probability that i is the prior state and j is the current state. In what follows, $F(x, p, \lambda)$ will denote the cumulative probability distribution function of a non-central Chi-square distribution with *p* degrees of freedom and non-centrality parameter $\lambda = n_i d^2$, where *d* is the amount of the process mean shift presented by Mahalanobis distance measure.

The p_{ij}'s are given as follows:

$$p_{11} = p_{21} = p_{31} = F(w, p, 0) \times e^{-\lambda h}$$
$$p_{12} = p_{22} = p_{32} = F(k, p, 0) \times e^{-\lambda h} - p_{11}$$
$$p_{13} = p_{23} = p_{33} = e^{-\lambda h} - p_{12} - p_{11}$$
$$p_{14} = F(w, p, n_1 d^2) \times (1 - e^{-\lambda h})$$
$$p_{15} = F(k, p, n_1 d^2) \times (1 - e^{-\lambda h}) - p_{14}$$
$$p_{16} = 1 - e^{-\lambda h} - p_{15} - p_{14}$$

$$p_{24} = p_{34} = F(w, p, n_2 d^2) \times (1 - e^{-a h})$$
$$p_{25} = p_{35} = F(k, p, n_2 d^2) \times (1 - e^{-a h}) - p_{24}$$
$$p_{26} = p_{36} = 1 - e^{-a h} - p_{25} - p_{24}$$
$$p_{44} = F(w, p, n_1 d^2)$$
$$p_{45} = F(k, p, n_1 d^2) - p_{44}$$
$$p_{46} = 1 - p_{45} - p_{44}$$
$$p_{54} = F(w, p, n_2 d^2)$$
$$p_{55} = F(k, p, n_2 d^2) - p_{54}$$
$$p_{56} = 1 - p_{55} - p_{54}$$

Now, upon the cost model specified by Costa and Rahim (2001), the expected loss per hour is determined by following function. In what follows, V_0 is the hourly profit earned when the process is in control, V_1 is the hourly profit earned when the process is out-of-control, C_0 is the expected cost for investigating false alarms, C_1 is the expected cost to find assignable cause and repair the process, s stands for the cost of each inspected item, T_0 determines the time needed to investigate a false alarm, T_1 is the time needed to search for and repair the assignable, $\mathbf{b}' = (0,\ 1,\ 0,\ 0,\ 0)$ is a vector of initial probabilities, $\mathbf{I}$ is the identity matrix of order 5, $\mathbf{Q}$ is the 5×5 matrix obtained from **P** on deleting the elements corresponding to the absorbing state, $\mathbf{h}' = (h,\ h,\ h,\ h,\ h)$ is the vector of sampling intervals, $\vec{\mathbf{1}}_F = (0,\ 0,\ 1,\ 0,\ 0)'$ and $\vec{\mathbf{n}} = (n_1,\ n_2,\ n_2,\ n_1,\ n_2)'$

$$E(A) = V_0 - \frac{\frac{V_0 - V_1}{a} + \vec{\mathbf{b}}'(\mathbf{I} - \mathbf{Q})^{-1}(V_1 \times \vec{\mathbf{h}} - C_0 \times \vec{\mathbf{1}}_F - s \times \vec{\mathbf{n}}) - C_1}{\vec{\mathbf{b}}'(\mathbf{I} - \mathbf{Q})^{-1}(\vec{\mathbf{h}} + T_0 \times \vec{\mathbf{1}}_F) + T_1} \qquad (1)$$

In order to economically design the VSS, we try to find five chart parameters (control limit *k*, warning line w, sample sizes n_1 and n_2 and sampling frequency) that minimize *E*(*L*). The cost function (1) has a discontinuous and non-convex solution space and hence is solved by using genetic algorithms (GAs). The crossover rate 0.1, the population size 100 and the mutation rate 0.5 are used here and the number of generation is set to at least 10,000 times.

3. COMPARISON RESULTS

Table 1 gives the thirteen process and cost parameters which is taken from Costa and Rahim (2001). These values provide a general variation in parameter values and shall be used here for our comparison purpose. Table 2 gives the optimal design parameters and resulting loss for the economic VSS scheme with a comparison to the economic FRS scheme for the cases *p*=2. The results are opposed to what Aparisi (1996) and Faraz and Moghadam (2008) achieved. They concluded that the VSS scheme detects the process mean shifts more quickly than the FRS scheme and hence reduce the resulting cost due to nonconforming items. Their result is somehow true but we show that when we take a look at the VSS scheme by considering complete economic information, it is not economical than the FRS scheme. In fact, the results indicate that the VSS scheme can be designed in such a way that is able to detect the process mean shifts more quickly than the FRS scheme, but clearly this method is not economical at all. Hence, the other VRS schemes like the VSI and the VSSVSI can be applied to achieve this goal.

Table 1 The 13 process and cost parameters

No.	s	C_0	C_1	V_0	V_1	T_0	T_1	a	d
1	5	500	500	500	50	5	1	0.01	**1**
2	**10**	500	500	500	50	5	1	0.01	**1**
3	5	**250**	500	500	50	5	1	0.01	**1**

4	5	500	**50**	500	50	5	1	0.01	**1**
5	5	500	500	**250**	50	5	1	0.01	**1**
6	5	500	500	500	**100**	5	1	0.01	**1**
7	5	500	500	500	**0**	5	1	0.01	**1**
8	5	500	500	500	50	**2.5**	1	0.01	**1**
9	5	500	500	500	50	5	**10**	0.01	**1**
10	5	500	500	500	50	5	1	**0.05**	**1**
11	5	500	500	500	50	5	1	0.01	**1.5**
12	5	500	500	500	50	5	1	0.01	**0.5**
13	**5**	**500**	**500**	**500**	**50**	**5**	**1**	**0.01**	**2.0**

Table 2 The optimal parameters of the economic design of the FRS and VSS schemes for the case p =2

No	*FRS scheme*						*VSS scheme*							
	k	*h*	*n*	*ANF*	*AATS*	*E(L)*	*k*	*w*	*h*	n_1	n_2	*ANF*	*AATS*	*E(L)*
1	10.49	6.27	18	0.08	4.08	43.37	10.46	4.96	5.86	16	20	0.08	4.05	43.23
2	9	8.37	16	0.13	5.45	54.62	9.08	4.39	8.12	15	18	0.12	5.42	54.49
3	10.31	6.22	18	0.09	4.00	43.17	10.44	5.22	6.04	17	21	0.09	3.98	43.05
4	10.51	6.24	18	0.08	4.07	39.10	10.64	5.12	6.06	17	21	0.08	4.05	38.97
5	9.32	9.12	16	0.10	6.05	28.54	9.42	4.2	8.87	15	18	0.1	6.01	28.48
6	10.5	6.68	18	0.08	4.35	41.38	10.64	5.14	6.49	17	21	0.08	4.32	41.27
7	10.5	5.94	15	0.09	3.87	45.25	10.64	5.14	5.76	14	18	0.09	3.84	45.11
8	9.43	5.97	16	0.14	4.00	42.22	9.49	4.72	5.78	15	18	0.14	3.98	42.1
9	10.36	6.53	18	0.08	4.22	79.24	10.49	5.2	6.33	17	21	0.07	4.2	79.13
10	10.03	3	17	0.04	2.01	114.23	10.2	4.68	3.03	17	19	0.04	2	114.1
11	12.22	4.44	9	0.05	2.89	33.52	12.49	6.02	4.46	9	10	0.04	2.9	33.38
12	7.46	11.06	52	0.21	7.59	68.95	7.46	0.00	11.06	25	52	0.21	7.59	68.95
13	13.42	3.46	5	0.03	2.07	28.30	13.62	6.64	3.47	5	6	0.02	2.06	28.24

4. CONCLUDING REMARKS

The Hotelling's T^2 control chart is a widely applied multivariate control chart for detecting moderate to large shifts in process mean. In the present paper, the economic design of the VSS T^2 control chart is developed based on the cost model considered by Costa and Rahim (2001). The expected total cost per hour is minimized using GA and finally a comparison is made between the FRS and VSS schemes from an economic viewpoint. The results indicate that the VSS scheme is not more economical than the FRS scheme and hence the VSS scheme is not preferred to the FRS scheme for monitoring purpose.

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