Own linear vibrations of cylindrical shells in elastic medium

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Article History
Received: 30 April 2019
Accepted: 21 June 2019
Published: June 2019

Citation
Muxitdinov RT. Own linear vibrations of cylindrical shells in elastic medium. Science & Technology, 2019, 5, 140-146

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ABSTRACT
In this paper, we consider the natural linear oscillations of cylindrical shells in an infinitely elastic medium. The frequency equation is obtained, which is solved by the methods of Muller. An analysis of the numerical results.

Keywords: Frequency, damping coefficient, natural frequencies, elastic medium, shells, oscillations.

1. INTRODUCTION
The ideal elastic body has no loss. Such a body is characterized by a linear one-to-one relationship between stress and strain during the entire period of alternating stress [1, 2]. It follows that stress and strain are always in phase. The dissipation of the energy of an elastic wave will occur if the stress and strain are not connected by a unique dependence during the oscillation period. The absence of such an unambiguous relationship between stress and strain arises when temporary derivatives of stress and (or) strain appear in the equation relating stress and strain. Even if the equation is linear with respect to stresses and strains, the presence of time derivatives is always associated with dissipation. As a result, at alternating voltage, a hysteresis effect occurs. This means that in the
frequency range in which the attenuation has a noticeable amount, the deformation lags behind the stress. The attenuation in the empirical model of a solid (standard linear viscoelastic body) is considered in detail in [1,2].

2. FORMULATION OF THE PROBLEM

In the case of a sufficient length of cylindrical shells and its environment, it is reduced to a plane problem of the dynamic theory of elasticity. Under the assumption of a generalized plane-deformed state, the equation of motion of the ambient mixing medium is of the form [3]

\[(\lambda + 2\mu) \nabla \text{div} \bar{u} - \mu \nabla \text{rot} \bar{u} + \vec{b} = \rho \frac{\partial^2 \bar{u}}{\partial t^2},\]  

where \(\lambda\) and \(\mu\) - elastic moduli called Lame constants; \(\vec{b}\) - volume force density vector \((b = 0)\); \(\rho\) - material density among, \(\bar{u}(u_\theta, u_r)\) - medium displacement vector;

\[\text{grad} \varphi = \frac{\partial \varphi}{\partial r} \vec{K}_r + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \vec{K}_\theta; \quad \text{div} \bar{u} = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta};\]

\(\vec{K}_r\) and \(\vec{K}_\theta\) - unit vectors, \(\bar{u} = u_r \vec{K}_r + u_\theta \vec{K}_\theta\) - environmental displacement vector. The problem is solved in displacement potentials

\[u_r = \frac{\partial \varphi}{\partial r} \frac{1}{r} \frac{\partial \psi}{\partial \theta}; \quad u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r}.\]

Potentials \(\varphi\) and \(\psi\) satisfy the wave equation

\[\nabla^2 \varphi = \frac{1}{C_\alpha^2} \frac{\partial^2 \varphi}{\partial t^2}; \quad \nabla^2 \psi = \frac{1}{C_\beta^2} \frac{\partial^2 \psi}{\partial t^2} \]  

The equation of motion of cylindrical shells in a flat formulation is:

\[\begin{align*}
\frac{\partial^2 u}{\partial \theta^2} &+ \frac{\partial W}{\partial \theta} = -\frac{R^2}{B} x_1 \\
\frac{\partial u}{\partial \theta} + b^2 \left( \frac{\partial^4 W}{\partial \theta^4} + 2 \frac{\partial^2 V}{\partial \theta^2} + W \right) + W &= \frac{R^2}{B} x_2
\end{align*}\]

\[x_1 = -\sigma_{r\theta} \bigg|_{r=R} - \rho_o h_o \frac{\partial^2 u}{\partial t^2}; \quad x_2 = -\sigma_{rr} \bigg|_{r=R} - \rho_o h_o \frac{\partial^2 W}{\partial t^2};\]

\[b^2 = \frac{h_o^2}{12 R^2} \quad B = \frac{E_o h_o}{1 - \nu_o^2};\]

\(R\) - shell radius, \(\rho_o\) - shell density, \(\nu_o\) - Poisson's ratio, \(E_o\) - shell elastic modulus, \(\sigma_{rr}\) and \(\sigma_{r\theta}\) - normal and tangential, constituting reactions from the environment, the contact between the shell and the environment can be hard or sliding:

\[u \bigg|_{r=R} = u_\theta \bigg|_{r=R}, W \bigg|_{r=R} = u_r \bigg|_{r=R}\]  

(4)
At natural oscillations of infinity, the shortened Sommerfeld conditions [3] are set, i.e.

\[ \lim_{r \to \infty} \left( \sqrt{r} \left( \frac{\partial \phi}{\partial r} + i K_1 \phi \right) \right) = 0 \quad \lim_{r \to \infty} \left( \sqrt{r} \left( \frac{\partial \psi}{\partial r} + i K_2 \psi \right) \right) = 0 \]

3. SOLUTION METHODS

The solution of the wave equation (3) is sought in the form

\[
\begin{pmatrix}
\phi \\
\psi
\end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix}
\phi_n(r) \\
\psi_n(r)
\end{pmatrix} \begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix} e^{-i \omega t},
\]

which \( \phi_n(r) \) and \( \psi_n(r) \) displacement potentials satisfy the Helmholtz equations:

\[ \nabla^2 \phi_n + K_n^2 \phi_n = 0, \nabla^2 \psi_n + K_n^2 \psi_n = 0 \quad K_n^2 = \frac{\omega^2}{c_n^2}, \quad (i = 1, 2) \]

\[ c_n^2 = (\lambda_i + 2\mu_i) / \rho_i, \quad c_2^2 = \mu_i / \rho_i. \]

where \( \omega \) - purity; \( K_n \) - number of waves; \( t \) - time;

The solution of the Helmholtz equation in cylindrical coordinates is expressed in terms of first-order and second-order Henkel functions of the nth order:

\[ \phi_n = A_{n1} H_n^{(1)}(K_1 r) + B_{n1} H_n^{(2)}(K_1 r), \quad \psi_n = A_{n2} H_n^{(1)}(K_2 r) + B_{n2} H_n^{(2)}(K_2 r), \]

where \( A_{n1} \) and \( B_{n1} \) - arbitrary constant which is determined from the boundary conditions (4), \( H_n^{(1)}(K_1 r) \), \( H_n^{(2)}(K_2 r) \) - Henkel functions of the 1st and 2nd kind of the nth order.

As a first example, consider the eigenmodes of a cylindrical hole located in an elastic medium. At the boundary \( r = R \), we set a voltage-free condition i.e.

\[ \sigma_{rr} \bigg|_{r=a} = \sigma_{r\theta} \bigg|_{r=a} = 0. \]

Substituting (6) into (7), we obtain the frequency equation

\[ Z_{1n} X_{2n} + Z_{2n} X_{1n} = 0 \]

where

\[ X_{1n} = \Omega_0 H_{n+1}^{(1)}(\Omega_o) + (a_{n2}^1 - d_1^1) \Omega_0^2 H_{n+1}^{(1)}(\Omega_o); \]

\[ X_{2n} = n[(n-1) H_{n+1}^{(1)}(\Omega_1) - \Omega_1 H_{n+1}^{(1)}(\Omega_1)]; \]

\[ Z_{1n} = n[(1-n) H_{n+1}^{(1)}(\Omega_o) - \Omega_o H_{n+1}^{(1)}(\Omega_o)]; \]

\[ Z_{2n} = (a_{n2}^1 - \Omega_1^2 / 2) H_{n+1}^{(1)}(\Omega_1) + \Omega_1 H_{n+1}^{(1)}(\Omega_1); \]

\[ d_1 = (1-v_1) / (1-2v_1); \quad a_{n2}^1 = n^2; \quad a_{n2}^1 = n^2 - n; \quad \Omega_o = \Omega_o L_1; \]

\[ l_1 = (1-2v_1) / (2(1-v_1)); \quad \Omega_1 = \omega a / C_{pl} \]

The solution of the wave equation (2) takes the following form
\[
\left( \phi \right) = \sum_{n=0}^{\infty} \left( A_{n1} H_n^{(1)}(K_1 r) + B_{n1} H_n^{(2)}(K_1 r) \right)
\]
\[
\left( \psi \right) = \sum_{n=0}^{\infty} \left( A_{n2} H_n^{(1)}(K_2 r) + B_{n2} H_n^{(2)}(K_2 r) \right).
\]

From the boundary conditions follows, what \( H_n^{(2)} (z) \) describes the descending wave, so solutions (8) will take the form

\[
\left( \phi \right) = \sum_{n=0}^{\infty} A_n H_n^{(1)}(ar),
\]
\[
\left( \psi \right) = \sum_{n=0}^{\infty} C_n H_n^{(1)}(br),
\]

After putting (9) into the boundary conditions (7), we obtain a system of algebraic equations with complex coefficients

\[
[D][q]=0,
\]

where \([q]=\{A_n, C_n\}\)-column vector of arbitrary constants; \([c]\) is a square matrix, whose elements are expressed in terms of Hankel functions of the first kind of the nth order. In order for a system of algebraic equations to have a non-trivial solution, it is necessary and sufficient

\[
[c]=0 \quad (10)
\]

The roots of the transcendental (10) equation describe the frequency of natural oscillations of the cavity. The frequency equation (10) takes the following form

\[
D_{\rho} = xH_{\rho-1}(\rho^2 - 1) yH_{\rho-1}(\rho^2 - \rho + y^2 / 2)H_{\rho}(y)
- yH_{\rho}(x)((\rho^3 - \rho + y^2 / 2) yH_{\rho}(y) - (\rho^2 - \rho - y^2 / 4) y^2 H_{\rho}(y)) \quad (11)
\]

where \( x = wa(p/(\lambda + 2\mu))^{1/2}; y = wa(p/\mu)^{1/2} \), \( \lambda \) and \( \mu \)- Lame coefficients;

\( p \)-the density of the material. Equation (11) after some transformations can be written as follows

\[
(\rho^2 - 1) F(x) F(y) - (y^2 / 2) F(x) + F(y) + \rho^2 - (\rho^2 - y^2 / 2)^2 = 0.
\]

where \( F(x) = xH_{\rho}^{(1)}(x)/H_{\rho}(x), \rho = 1,2,3,..... \)

Let us consider radial oscillations of a spherical cavity in an unbounded medium, accompanied by the emission of longitudinal sound waves, which leads to a loss of energy, and thus to the damping of oscillations. With \( C_p >> C_S \) The considered problem is equivalent to the problem of natural oscillations of a spherical body. The roots of the characteristic equation (11) are found by the Muller method [4]. Based on the above studies, it was revealed that the mechanical system under consideration has a discrete complex Eigen frequency. Table 1 shows the results obtained and their comparison, the results of those of other authors [5,6,15-21].

The results show that with an increase in the elastic modulus, the corresponding natural frequencies of the mechanical system slowly increase.

Define \( \Omega \) with different Poisson's ratios \( \nu_1 \) and \( n \). With \( n=0 \) we obtain axis symmetric oscillations of a cylindrical hole. The partial equation (10) takes the form

\[
-\Omega d_{1}H_{\rho}^{(1)}(\Omega_1) + H_{1}^{(1)}(\Omega_1) = 0. \quad (12)
\]

<table>
<thead>
<tr>
<th>№</th>
<th>Our results</th>
<th>Pao and Mao</th>
<th>Brnon and Parnes[2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.44741-0.44420 i</td>
<td>0.44647-0.44127 i</td>
<td>0.4464-0.4410 i</td>
</tr>
<tr>
<td>1</td>
<td>1.09272-0.77653 i</td>
<td>1.09272-0.7653 i</td>
<td>1.0929-0.441 i</td>
</tr>
<tr>
<td>2</td>
<td>1.90755-0.89782 i</td>
<td>1.90754-0.8978 i</td>
<td>1.9076-0.897 i</td>
</tr>
</tbody>
</table>
Frequency equation (11) is solved numerically, i.e. Muller method. Calculation results \( n \geq 0 \) \( (v_1 = 0.25) \) natural oscillations are shown in Table 1. As can be seen from the table, the corresponding complex frequencies increase with increasing number of waves around the circumference. Complex frequencies consist of two parts, real \((\text{Re} \Omega)\) and imaginary parts \((\text{Im} \Omega)\) which means the natural frequencies and damping coefficients. Frequency equations (12) depends only on the parameter \( \nu \) (Poisson's ratio).

With increasing Poisson's ratio within \( 0 \leq \nu \leq 0.4 \) real and imaginary parts of the complex frequency changes to 27%. With \( \nu = 0.5 \) the environment becomes incompressible, the attenuation is naturally absent. To check the obtained results are compared with the results of work [7, 8, 9-14] with \( \nu = 0.25 \).

Now consider the natural oscillations of a rigid cylindrical inclusion. In this case, we look for the solution of the wave equation and the hard inclusion in the form. At the contact \( r = a \) we set the condition of hard contact. The partial equation with \( n = 1 \) takes the form

\[
\Delta(\Omega_1) = 4\eta H_1(\Omega_1)H_1(\Omega_2) - \left(1 - \eta\right)\Omega_2 H_o(\Omega_2)H_1(\Omega_1) - (1 + \eta) \Omega_1 H_o(\Omega_2)H_1(\Omega_2) + \Omega_1 \Omega_2 H_o(\Omega_1)H_o(\Omega_2);
\]

\[
\Omega_1 = \alpha_1 a; \quad \Omega_2 = \beta_1 a; \quad \text{where} \quad \eta = \rho_1 / \rho_2; \quad \rho_2 - \text{hard inclusion density}; \quad \Omega_1 = \alpha_1 a; \quad \Omega_2 = \beta_1 a;
\]

The results of the calculations are presented in the table. 2 \( (v_1 = 0.25) \), according to which \( \eta \geq 1 \) real parts of complex intrinsic purity vanish.

| Table 2 The dependence of the complex natural frequencies of the cylindrical hole |
|---|---|---|---|
| \( n=0 \) | \( n=1 \) | \( n=2 \) | \( n=3 \) |
| 0.4529D+00 | 0.10927D+01 | 0.19075D+01 | 0.27565D+01 |
| -10.47651D+00 | -10.76538D+00 | -10.89782D+00 | -10.99155D+00 |
| 0.28621D+00 | 0.17852D+00 | 0.72325D+01 | 0.32283D+01 |
| -10.17852D+00 | -10.404607D+00 | -10.12307D+00 | -10.22283D+00 |

At \( \eta = 0 \) we get environmental fluctuations around a rigid body, i.e. we have only imaginary roots. As a result of using the asymptotic value of the Henkel function (for \( a >> l \)), we obtained

\[
\omega = -i(C_{s1} / C_{p1} + 1)C_{s1} / a .
\]

The existence of an imaginary value of the natural frequency means that the oscillatory processes in the system are only damped. The imaginary natural frequencies turn out to depend on the longitudinal and transverse speeds, as well as the hole radius. The existence of a discrete frequency plays an important role for the calculation of underground pipelines located in the ground environment. The obtained numerical results are presented in the form of tables and figures. The appearance of an additional free surface mainly thickens and reduces the eigenvalue frequency by 10-16%. The existence of a natural frequency means that Rayleigh waves can occur in the vicinity of the free surface of a cylindrical hole. Thus, according to (11) with \( p \to 0 \) the real part of the complex frequency does not exist.

In the second example, we consider natural oscillations of cylindrical shells in an elastic medium. The solution of equations (1) and (3) is sought in the form
\[
\begin{pmatrix}
V \\
W \\
u_\theta \\
u_r
\end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix}
V_n(R) \\
W_n(R) \\
U_\theta(r) \\
U_r(r)
\end{pmatrix} \begin{pmatrix}
\sin n\theta \\
\cos n\theta \\
\sin n\theta \\
\cos n\theta
\end{pmatrix} e^{-i\alpha r} 
\]

(14)

Table 3 The change in complex frequency, depending on \( \tilde{E} \) (\( E = E_1 / E_0 \)) at \( \eta = 4; \gamma_0 = \nu = 0.14 \) (hard contact)

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega_1 )</td>
<td>2.2642BD-01</td>
<td>1.8172BD-01</td>
<td>1.5839BD-01</td>
<td>1.4324BD-02</td>
</tr>
<tr>
<td>( \Omega_1 )</td>
<td>-1.2969BD-01</td>
<td>-1.7092BD-02</td>
<td>-1.5765BD-02</td>
<td>-1.4703BD-02</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>3.2339BD-01</td>
<td>2.3924BD-01</td>
<td>2.0207BD-01</td>
<td>1.7995BD-01</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>-2.5641BD-01</td>
<td>-2.0152BD-02</td>
<td>-1.7192BD-02</td>
<td>-1.5278BD-01</td>
</tr>
<tr>
<td>( \Omega_3 )</td>
<td>4.8155BD+00</td>
<td>4.8144BD+00</td>
<td>4.8137BD+00</td>
<td>4.8134BD+00</td>
</tr>
<tr>
<td>( \Omega_3 )</td>
<td>4.7709BD+00</td>
<td>4.7677BD+00</td>
<td>4.7667BD+00</td>
<td>4.7662BD-02</td>
</tr>
</tbody>
</table>

where \( V_n(R), W_n(R), U_\theta(r) \) and \( U_r(r) \) - displacement amplitude, \( \omega = \omega_R + i\omega_I \) - complex natural frequency The results of the calculations are presented in table 3.

As we see, \( (E_1 / E_0) \geq 0.21 \) the real parts of the natural frequency vanish, and the behavior of the imaginary parts remains unchanged.

Get out.

The problem statement is proposed for the natural oscillations of cylindrical bodies in a deformable medium. The task is to find those \( \Omega = \Omega_R + i\Omega_I \) (\( \Omega_R \) - real and \( \Omega_I \) - imaginary parts of complex Eigen frequencies), in which the system of equations of motion and shortened radiation conditions have a nonzero solution in the class of infinitely differentiable functions. It is shown that the problem has a discrete spectrum.

**REFERENCE**


