ANALYSIS VOL. 2, 2016

ISSN 2394–3750 EISSN 2394–3769

SCIENCE & TECHNOLOGY

On varying time-step in the analytical solution of black-scholes option price model (BSOPM) with and without a drift

Ganiyu AA✉, Lawal MO, Adenegan KE

Department of Mathematics, Adeyemi College of Education, Ondo. Nigeria

✉**Corresponding Author:**
Department of Mathematics, Adeyemi College of Education, Ondo. Nigeria; Email: ganiyuiwajowa@gmail.com

Article History
Received: 19 January 2016
Accepted: 23 February 2016
Published: April-June 2016

Citation
Ganiyu AA, Lawal MO, Adenegan KE. On varying time-step in the analytical solution of black-scholes option price model (BSOPM) with and without a drift. *Science & Technology*, 2016, 2(6), 91-99

ABSTRACT

This paper examines the solution of Black-Scholes option Price model (BSOPM) with and without a drift with due consideration to varying time-step. Each of the models is in the form of first order stochastic differential equations (SDEs). The authors compare the analytical solution of the two models numerically by discritisation of Wieners sample path within a given interval [0,1] using varying time-steps 1/10, 1/20, 1/30, 1/40 and 1/50. The results shows that for each time step the analytical solution of BSOPM with a drift is higher than the corresponding analytical solution of the BSOPM without a drift. Also, the mean of the exact solution (MES) shows that the MES of the BSOPM with drift function is greater than MES of BSOPM without a drift function. The graphical solution of each model using the varying time steps were drawn which clearly showed the disparity between the analytical solutions.

Keywords: Black-Scholes Option Price Model, Stochastic Differential Equations, Wiener sample Path, Gaussian White noise, Mean exact Solution.

1. INTRODUCTION

Stochastic Differential Equations (SDEs) are differential equations in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process (Vajargah and Asghari, 2014, Ganiyu et al, 2015). SDEs play vital roles in innumerable areas of study. Some of these areas are finance, electronics, physics, Biology, aerospace, neuroscience, etc. SDEs are used to model diverse phenomena such as fluctuating stock prices or physical system subject to thermal fluctuations.

The numerical methods for solving these equations show low accuracy especially for the case of high non-linear drift terms. It is therefore very important to search and present exact solutions for SDEs. The resulting solutions are also important to check for the accuracy of the existing numerical methods (Skiadas, 2010).

1.1. Aims and objectives

The aim of this paper is to determine the analytical solution of first order stochastic differential equations computationally. One of the objectives is to determine the analytical solution of two models used in finance computationally. These are Black-Scholes option price model with a drift function and Black-Scholes option price model without a drift function. Another objective is to compare the results obtained using mean of the exact solution. The MES of the k^{th} is defined by

$$MXeactk = \frac{\sum_{j=1}^{n} Xexactk_j}{n}, \quad k = 1, 2, \ldots \qquad (1)$$

Where n is the maximum number of iteration considered, and $Xeactk$ the exact solution of the k^{th} model . For example, when $k = 1$ and $n = 10$, the mean exact solution of the 1^{st} model is defined by

$$MXexact1 = \frac{\sum_{j=1}^{10} exact1_j}{10} \qquad (2)$$

where $exact1_1, exact1_2, \cdots, exact1_{10}$ represent the exact solution of the Black-Scholes option price model with a drift function for the 1st, 2nd , ..., 10th iteration.

Also, when $k = 2$ and $n = 10$, the mean exact solution of the 2^{nd} model is defined by

$$MXexact2 = \frac{\sum_{j=1}^{10} exact2_j}{10} \qquad (3)$$

where $exact2_1, exact2_2, \cdots exact2_{10}$ represent the exact solution of the Black-Scholes option price model without a drift function for the 1st, 2nd , ..., 10th iteration. Also, graphical solution of each model will be drawn to show the disparity between the solutions of each model relative to the time step used.

1.2. Statement of the problem for SDEs

In this paper we will consider a one dimensional Itô SDE of the form

$$dX = f\left(t, X(t)\right)dt + g\left(t, X(t)\right)\cdot dW, \quad X(t_0) = X_0, t \in [0, T] \qquad (4)$$

where $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is the deterministic or drift function, $g:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}\times\mathbb{R}^n$ is the diffusion function and $\gamma:=\{\gamma(t),t\in T\subseteq\mathbb{R}\}$ is Gaussian white noise. This is defined by $\gamma=\frac{dW(t)}{dt}$, $W(t)$ is the Wiener process or (Brownian motion).

It should be noted that the noise $W(t)$ is stochastic process. A stochastic process $\{X(t),t\in T\}$ or $(or\ \{X_t,t\in T\})$ is defined by Kannan (1979) as a family of random variables indexed by the parameter set T.

The following researchers have worked on stochastic differential equation: Kloeden et al (1991, 1995), Kamposky et al (1992), Penski (2000), Higham (2001), Higham et al (2002), Burrage (2004), Chummei (2012), Sauer (2013) and Fadugba et al (2013), Akinbo et al (2015), Kayode and Ganiyu (2015), Kayode et al (2016).

Every Wiener process according to Williams (2006) is either discrete or continuous-time stochastic processes. Williams (2006) in Ganiyu et al (2015) categorized the properties of Wiener process as follows:

(i) $W(t)$ is continuous and $W(0)=0$ with probability 1.

(ii) $W(t)\sim N(0,t)$ i.e. $W(t)$ is normally distributed with mean 0 and variance t.

(iii) $W(t)$ has independent increment, $W(t+s)-W(s)\sim N(0,t)$ and is independent of the history of the process up to time t.

(iv) Cov $((W(s),W(t))=\min(s,t)$

(v) Interprete $dW(t)=W(t+dt)-W(t)$.

Two problems in the form of Itô stochastic differential equation (4) will be considered. The first is the Black-Scholes option price model with a drift function. This is given by

$$\left.\begin{aligned}dX(t)&=\mu X(t)dt+\sigma X(t)dW(t)\\X(0)&=X_0\end{aligned}\right\}\tag{5}$$

Where $f(t,X(t))=\mu X(t)$ and $g(t,X(t))=\sigma X(t)$.

The analytical solution of the model in (5) was obtained by Ganiyu et al (2015) using Itô formula as

$$X(t)=X_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W(t)\right)\tag{6}$$

The second is Black-Scholes option price model without a drift function. This is given by

$$\left.\begin{aligned}dX(t)&=\sigma X(t)dW(t)\\X(0)&=X_0\end{aligned}\right\}\tag{7}$$

Where $f(t,X(t))=0$ and $g(t,X(t))=\sigma X(t)$.

The analytical solution of the model in (7) was obtained by Ganiyu et al (2015) using Itô formula as

$$\Rightarrow X(t) = X_0 \exp\left(-0.5\sigma^2 \cdot t + \sigma \cdot W(t)\right). \quad (8))$$

2. RESEARCH METHODOLOGY

The comparison of the analytical solution of the Black-Scholes option price model with a drift in equation (5) and without a drift in equation (7) will be obtained numerically by discritising Wiener sample path $W(t)$ with a given interval $[0,1]$ using varying time steps $dt = 1/10$, $1/20$, $1/30$, $1/40$, and $1/50$, where the number of time-step $N = 10$, 20, 30, 40 and 50. The constants of the drift and diffusion function will be taken as $\mu = 0.001$ and $\sigma = 0.002$ respectively for the case when the BSOPM is with a drift function. The constants of the drift and diffusion function will be taken as $\mu = 0$ and $\sigma = 0.002$ for the case when the BSOPM is without a drift function. The same models were considered by Ganiyu et al (2015). The difference between the results obtained in this paper and that of Ganiyu et al is that Ganiyu et al considered a single time-step $dt = 1/10$ to obtain the solution of the two models.

In this paper we are considering varying time-step. The Brownian increments dW will be computed in MATLAB as $dW = sqrt(dt)*rand(1, N)$ and discritised Wieners sample path W will be computed as $W = \mathrm{Cumsum}(dW)$. It should be noted that the computational results obtained depend on the analytical solution of the two models stated in (5) and (7). They have been obtained using Itô formulae. The method of solution can be seen in (Ganiyu et al, 2015).

3. RESULTS

3. 1. Computational Comparison of the Analytical Solution of Black-Scholes Option Price Model with and without a Drift Function Using Varying Time-Step

To compare the computed results for the analytical solution of BSOPM with a drift function and the analytical solution of BSOPM without a drift function, the results for BSOPM with a drift function will be denoted by Xexact1 while that for BSOPM without a drift function will be denoted by Xexact2. The results can be identified in tables 1, 2, 3, 4 and 5. The graphical solution for each model using the varying time step will also be obtained.

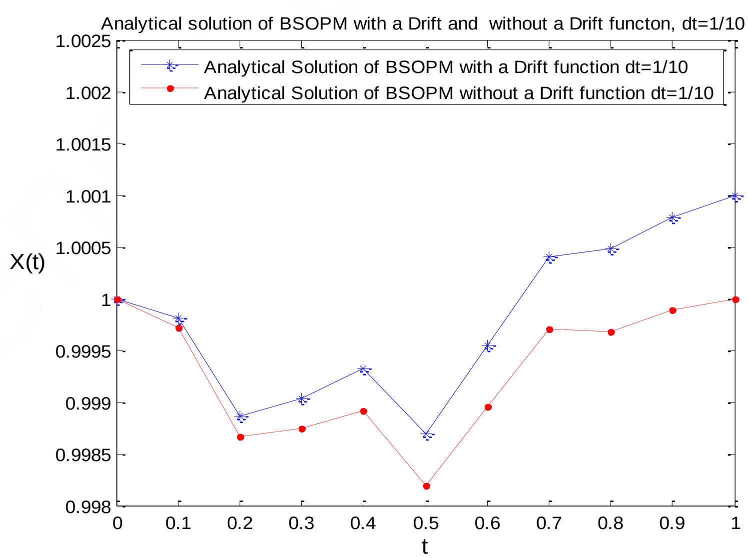


Table 1 Computational Comparison of Analytical Solution of BSOPM with and without a Drift function Using Time-Step dt=1/10

t-value	Xexact1	Xexact2
0.100000	0.999826237090509	0.999726259465765
0.200000	0.998873249198816	0.998673494525110
0.300000	0.999052130560785	0.998752459874467

0.400000	0.999333645704664	0.998933992182415
0.500000	0.998708965546868	0.998209735881912
0.600000	0.999561228806585	0.998961671954343
0.700000	1.000413111303150	0.999713067169270
0.800000	1.000489144214234	0.999689072970030
0.900000	1.000796139237500	0.999895827913053
1.000000	1.001006580240942	1.000006073997198

$MXexact1$ = 0.999806043190405, $MXexact1$ = 0.999256165593356

Table 1 shows the computational comparison of the analytical solution of BSOPM with a drift function represented by Xexact1 and that for BSOPM without a drift function represented by Xexact2 using time-step 1/10.

Table 2 Computational Comparison of Analytical Solution of BSOPM with and without a Drift function Using Time-Step dt=1/20.

t-value	Xexact1	Xexact2
0.100000	0.999161830617309	0.999061919429890
0.200000	0.999446135714132	0.999246266474579
0.300000	0.999565752599807	0.999265927849988
0.400000	1.000180454852443	0.999780462674271
0.500000	1.000504836552227	1.000004709176214
0.600000	1.000845951566830	1.000245624112136
0.700000	1.001660017480480	1.000959100817696
0.800000	1.001749924046653	1.000948844581926
0.900000	1.002354547038356	1.001452833777855
1.000000	1.002038641381670	1.001037103592645

$MXexact1$ = 1.000750809184991, $MXexact2$ = 1.000200279248720

Table 2 shows the computational comparison of the analytical solution of BSOPM with a drift function represented by Xexact1 and that for BSOPM without a drift function represented by Xexact2 using time-step 1/20.

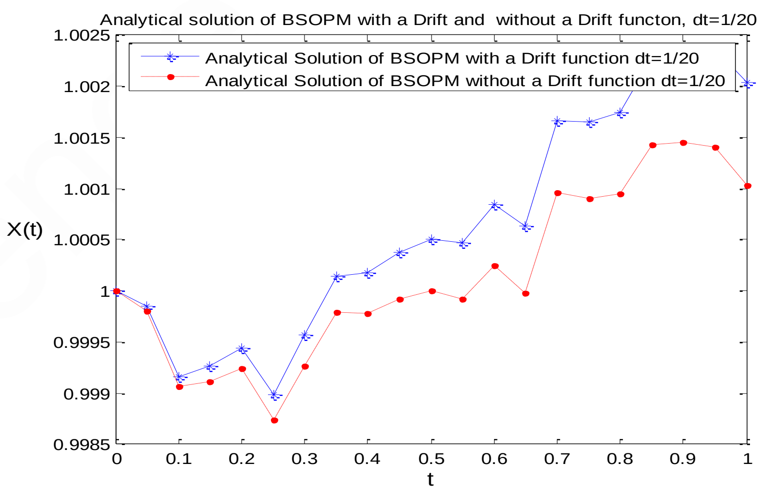


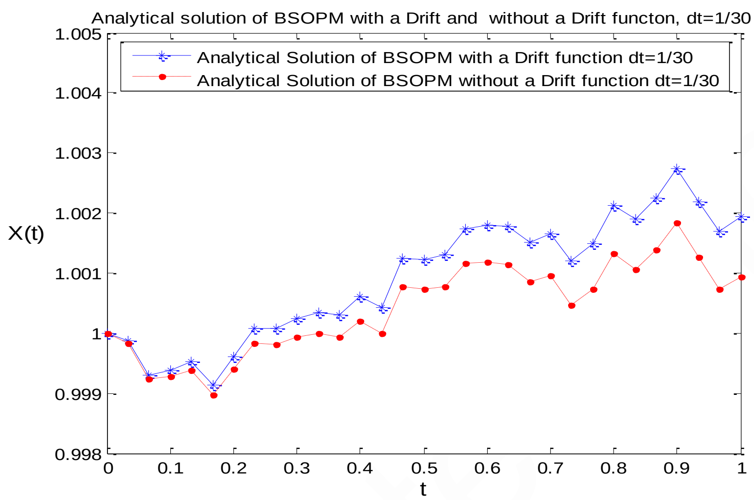
Table 3 Computational Comparison of Analytical Solution of BSOPM with and without a Drift function Using Time-Step dt=1/30.

t-value	Xexact1	Xexact2
0.100000	0.999379621641742	0.999279688676309
0.200000	0.999600582188674	0.999400682062915
0.300000	1.000240321187686	0.999940294097643
0.400000	1.000600886877804	1.000200726560452

0.500000	1.001233826139774	1.000733334360076
0.600000	1.001787237848709	1.001186345791644
0.700000	1.001655458119305	1.000954544646958
0.800000	1.002121908288143	1.001320531355026
0.900000	1.002741804002631	1.001839742367654
1.000000	1.001940171866060	1.000938732497332

$MXexact1$ = 1.001130181816053, $MXexact2$ = 1.000579462241601

Table 3 shows the computational comparison of the analytical solution of BSOPM with a drift function represented by Xexact1 and that for BSOPM without a drift function represented by Xexact2 using time-step 1/30.



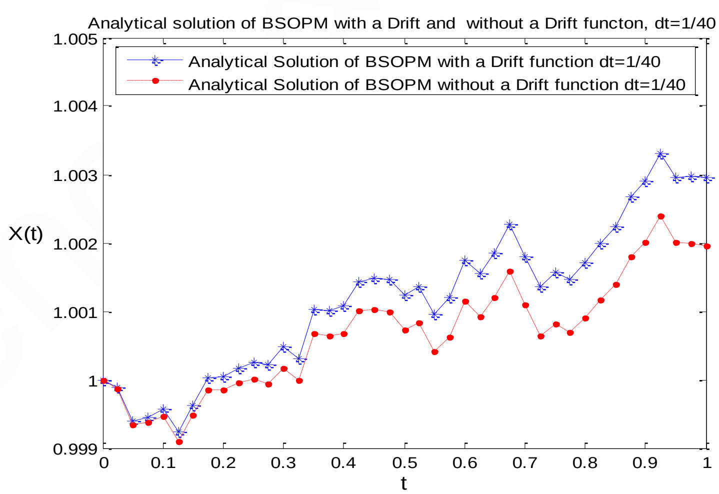


Table 4 Computational Comparison of Analytical Solution of BSOPM with and without a Drift function Using Time-Step dt=1/40.

t-value	Xexact1	Xexact2
0.100000	0.999567005565141	0.999467053862253
0.200000	1.000044913319673	0.999844924336574
0.300000	1.000474021989575	1.000173924799807
0.400000	1.001071521373238	1.000671172839733
0.500000	1.001234138105553	1.000733646169912
0.600000	1.001744564901602	1.001143698440625

0.700000	1.001789576510892	1.001088569188522
0.800000	1.001705911720997	1.000904867452050
0.900000	1.002910977431620	1.002008763609051
1.000000	1.002951404953899	1.001948954857531

$MXexact1$ = 1.001349403587219, $MXexact2$ = 1.000798557555606

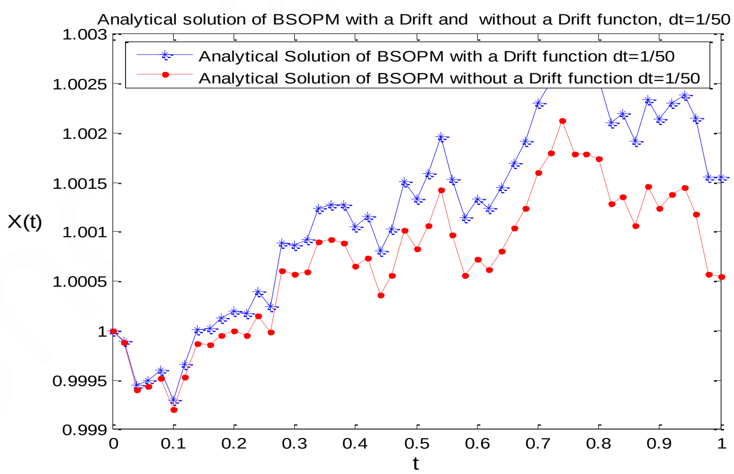
Table 4 shows the computational comparison of the analytical solution of BSOPM with a drift function represented by Xexact1 and that for BSOPM without a drift function represented by Xexact2 using time-step 1/40.

Table 5 Computational Comparison of Analytical Solution of BSOPM with and without a Drift function Using Time-Step dt=1/50.

t-value	Xexact1	Xexact2
0.100000	0.999299144947677	0.999199220029512
0.200000	1.000203231441779	1.000003210798222
0.300000	1.000868381595335	1.000568166115430
0.400000	1.001056604734918	1.000656262166875
0.500000	1.001327660612012	1.000827121926805
0.600000	1.001328028227518	1.000727411613584
0.700000	1.002308652672085	1.001607282123546
0.800000	1.002544923403436	1.001743208193555
0.900000	1.002136575351103	1.001235058176868
1.000000	1.001555435455315	1.000554380630694

$MXexact1$ = 1.001262863844118, $MXexact2$ = 1.000712132177509

Table 5 shows the computational comparison of the analytical solution of BSOPM with a drift function represented by Xexact1 and that for BSOPM without a drift function represented by Xexact2 using time-step 1/50.



3.2. Table of the Computational Comparison of the Mean Exact Solution (MES) of BSOPM with a Drift function and without a Drift Function Using Varying Time-Step dt.

dt-value	Mean of Xexact1	Mean of Xexact2
1/10	0.999806043190405	0.999256165593356
1/20	1.000750809184991	1.000200279248720
1/30	1.001130181816053	1.000579462241601
1/40	1.001349403587219	1.000798557555606
1/50	1.001262863844118	1.000712132177509

4. DISCUSSION

In this paper, the computational approach to analytical solution of first order stochastic differential equation (SDE) has been examined. Two models used in finance which appeared in the form of first order stochastic differential equations (SDEs) were used as test problems. These are Black-Scholes option price (BSOPM) with and without a drift function. The exact solution for each model was obtained computationally by using varying time-steps. This provides the opportunity for comparison of the results for the two models using mean exact solution (MES). The graphical solutions of the results for the two models were also drawn using varying time-step.

5. CONCLUSION

In this paper, two models which appeared in the form of first order SDEs have been considered. They are BSOPM with and without a drift function. The results were being generated using Itô formula and discritised Wieners sample path as tools. In obtaining the results, varying time- steps 1/10, 1/20, 1/30, 1/40, 1/50 were considered. The results showed that choosing each time-step, the analytical solution of BSOPM with a drift function is higher than the corresponding analytical solution of BSOPM without a drift function. Also, the mean of the exact solution (MES) calculated showed that the MES of BSOPM with a drift function is greater than the corresponding MES for BSOPM without a drift function for each of the time-step. The graphical solution can also be used to identify the disparity in computational analytical solution of each model for each time-step.

REFERENCE

1. Akinbo B.J., Faniran T. and Ayoola E.O. (2015). Numerical Solution of Stochastic Differential Equations. *International Journal of Advanced Research in Science, Engineering and Technology*. Vol. 2 (1), 608-616.
2. Burrage, K. et al (2004). Numerical methods for strong solutions of Stochastic differential Equations: An overview. Proceedings: *Mathematical Physical and engineering Science*, Published by Royal Society, 460(2041), 373-402.
3. Chunmei, S. Xiao, Y. and Zhang, C. (2012). "The convergence and MS stability of exponential Euler method for semilinear stochastic differential equations. Hindawi Publishing Corporation, *Abstract and Applied Analysis,* Volume 2012, Pp 1-19.
4. Fadugba, S.E., Adegboyegun, B.J., and Ogunbiyi, O.T. (2013). On convergence of Euler-Maruyama and Milstein scheme for solution of stochastic differential equations. *International journal of Applied mathematics and Modeling@ KINDI PUBLICATIONS.* Vol. 1, No 1, pp 9-15. ISSN: 2336-0054.
5. Ganiyu A.A., Olademo J.O. and Fakunle I. (2015). "On The Analytical Solution of BlackScholes Option Price Model (BSOPM) with and with No Drift". *ERJANSS Research Journal*, Achievers University, Owo, Ondo State. 1(1), 208-220.
6. Higham, D.J. (2001). An AlgorithmicIntroduction to Numerical Simulation of Stochastic Differential Equations. *SIAM Review,* 43(3), 525-546.
7. Higham, D.J., Mao, X. and Stuart, A.M. (2002). Strong convergence of Euler type methods fornon-linear stochastic differential equations. *SIAM Journal of Analysis.* (40), 1041-1063.
8. Kampowsky, W., Retrop, P. and Schmidt, W. (1992). Classification and numerical simulation of electrical circuits, surveys Math. indst, 2, 23-65.
9. Kannan, D. (1997). "An Introduction to Stochastic Processes". North Holland Series in Probability and Applied Mathematics. Pp 195-226.
10. Kayode S.J. and Ganiyu A.A. (2015). "Effect of Varying Stepsize in Numerical Approximation of Stochastic Differential Equations Using One Step Milstein Method. *Applied and Computational mathematics* Vol. 4(5) 351-362.
11. Kayode S.J., Ganiyu A.A. and Ajboye A.S. (2016). "One Step Method of Euler-Maruyama Type Effect of for Solution of Stochastic Differential Equations Using Varying Stepsizes". *Open Access Library Journal*, 3: 2247. http://dx.doi.org/10.4236/oalib.1102247.
12. Kloeden, P.E. and Platen, E. (1991). Stratonovich and Itô Taylor expansions, *Math.Nachr*. 151, 33-50.
13. Kloeden, P.E. and Platen, E. (1995). Numerical solution of Stochastic Differential Equations, ISBN 3-540-54062-8, Springer-Verlag, Berlin Heidelberge, New York.
14. Oksendal, B. (1998). Stochastic Differential Equations. An Introduction with Application, fifth Edition. ISBN 3-540-04758-1 Springer-Verlag, Berlin, Heidelberge, Italy. Pp 64-67.
15. Penski, C. (2000). A new numerical method for Stochastic differential Equation and its application in circuit simulation, *Com. and Appli.maths*, 115, 461-570.
16. Rawat, T.K., Parthasarathy, H. (2008). Modeling of an RC circuit using Stochastic Differential Equations, *int. j.Sc.Tech, vol.13. No. 2. Computation, School of Informatics,* University of Edinburgh.
17. Sauer, T. (2013). "Computational Solution of Stochastic Differential Equations. WIREs *Compt Stat* 2013.doi: 10.1002/wics.1272.
18. Skiadas, C.H. (2010). Exact Solution of Stochastic Differential Equations: Gampertz, Generalized Logistic and Revised

Exponential. *Method Comput Appl Probab* 12: 261-270. Doi 10.1007 /s110 09-009-9145-3.

19. Vajarghah, B.F. and Asghari, R. (2014). Simulation Method for Solving Stochastic Differential Equations with Constant Coefficients. *Journal of Mathematics and Computer Science*. 8, 28-32. Journal HomePage: www.tjmcs.com.
20. Williams, C. (2006). A tutorial Introduction to Stochastic Differential Equations: Continuous-time Gaussian Markov Process. Institute for Adaptive and Neural Computation, School of Informatics, University of Edinburgh, UK. Home Pages.