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Quantitative model of single-server queue system

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ABSTRACT

This study considered the model of queuing system. Examples of queue systems are Single-server queue system which occur if arrival and service rate is Poisson distributed (single queue) (M/M/1) queue; Multi-server queue system which comprises of single queue, many servers (M/M/c) queue hence Poisson servers. This study concentrates more on the single-server queue system. Single-server queue system is modelled based on Poisson Process with the introduction of Laplace Transform. PASTA was introduced in queuing systems with Poisson arrivals. Finally, the essence of queuing modelling is to ensure maximum utilization of a system at a minimized constraint. The main objective of a waiting system is to ensure that service rate is greater than the arrival rate such that the system will be regarded as efficient.

1. INTRODUCTION

The concept of queue was first used for the analysis of telephone call traffic in 1913 (Copper, 1981; Gross and Harris, 1985; Bastani, 2009). In a system that deals with the rate of arrival and service rate, waiting time is inevitable and it is always influenced by queue length. It is therefore crucial to minimize the waiting time to the lowest level in the bus terminal (Jain, Mohanty and Bohm, 2007). Using a petrol or gas station as an example, the queue length is influenced by how often a vehicle arrives at the station to be serviced. This is referred to as queuing system. Therefore it is pertinent to understand how queuing system works toward solving a particular problem.

Queuing theory is concerned with the issue of waiting; it is important to note that waiting can be quite boring, hence queuing theory examines how customers (vehicles) arrive to receive service by servers which is between arrivals of vehicles, start of service, and wait in queue. The basic application of queue is shown in Figure 1, also the basic quantities are:

i. Number of customers in queue L (for length);
ii. Time spent in queue W for (wait)

Figure1 Basic application of queue

Queue is applicable in transport management, communication management, congestion management, resource management, physical distribution management, logistics management, customers' satisfaction and service quality. Examples of queue system are:

1. Single-server queue system: This is also referred to as single queue, single server. It is simple if arrivals and services are Poisson distributed (M/M/1) queue. It has limited number of spots and not difficult. Figure 2 depicts single-server queue system.

Figure 2 Single-server queue system

2. Multi-server queue system: This is comprises of single queue, many servers (M/M/c) queue. The c is referred to as Poisson servers. Figure 3 depicts multiple-server queue system.

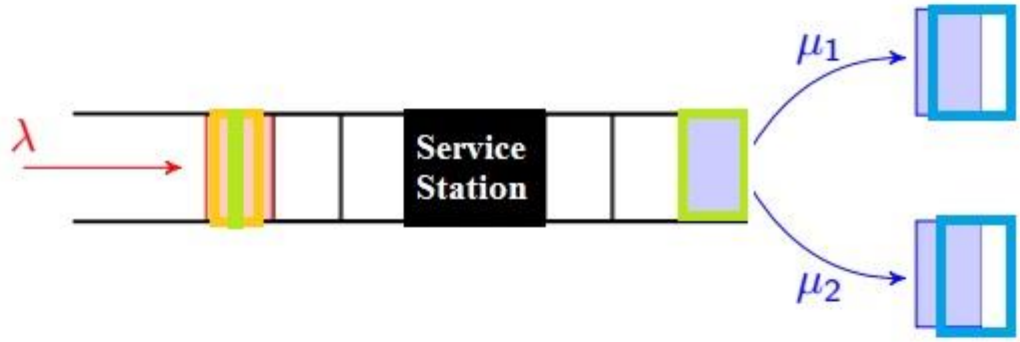

Figure 3 Multiple-server queue system

In single-server queue system, arrival and service processes are Poisson such that

a. Customers arrive at an average rate of λ per unit time;
b. Customers are serviced at an average rate of μ per unit time;
c. Interarrival and inter-service time are exponential and independent;
d. Hypothesis of Poisson arrivals is reasonable; and
e. Hypothesis of exponential service times are not so reasonable

In order to explain how the queueing system works, there is need to first introduce the *Poisson Process* (PP). It has exceptional properties and is a very important process in queuing theory. To simplify the model, we often assume customer arrivals follow a PP. The *Laplace Transform* (LT) is also a very powerful tool that was adopted in the analysis. Apart from PP and LT, there is focus on the queue model itself (Trani, 2011).

2. MODELLING OF SINGLE QUEUE SYSTEM

Queueing System from Poisson Process and "PASTA"

As earlier mentioned, the PP is important in queue theory due to its outstanding properties. According to Adan and Resing (2015), queueing system is achieved as "let N(t) be the number of arrivals in [0, t] for a PP with rate λ, i.e. the time between successive arrivals is exponentially distributed with parameter λ and independent of the past. Then N(t) has a Poisson distribution with parameter λt, as a result

$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, for k = 0, 1, 2.................... *Equation 1*

The mean, and coefficient of variation of N(t) are

Mean: $E(N(t)) = \lambda t$;

Coefficient of Variation: $c^2 N(t) = \frac{1}{\lambda t}$ .... *Equation 2*

By the memory less property of Poisson distribution, it can be verified that

$P(\text{arrival in } (t, t + \Delta t]) = \lambda \Delta t + 0(\Delta t)$ .... *Equation 3*

Hence, when Δt is small,

$P(\text{arrival in } (t, t + \Delta t)) \approx \lambda \Delta t$ .... *Equation 4*

In each small time interval of length Δt the occurrence of an arrival is equally likely. In other words, Poisson arrivals occur completely randomly in time. The Poisson Process is an extremely useful process for modelling purposes in many practical applications. An important property of the Poisson Process is called "PASTA" (Poisson Arrivals See Time Averages).

PASTA is meant for queuing systems with Poisson arrivals, (M/./. systems), arriving vehicles find on average the same situation in the queuing system as an outside observer looking at the system at an arbitrary point in time. More precisely, the fraction of vehicles finding on arrival the system in some *state A* is exactly the same as the fraction of time the system is in *state A*.

Laplace Transform

The Laplace transform $L_X(s)$ of a nonnegative random variable X with distribution function f(x) is define as:

$L_X(s) = E(e^{-sX}) = \int_{x=0}^{\infty} e^{-sX} f(x)dx$ .... *Equation 5*

It can be noted that

$L_X(0) = E(e^{-X.0}) = E(1) = 1$ .... *Equation 6*

and

$L^1_X(0) = E((e^{-sX})^1)|s=0$

$= E(-Xe^{-sX})|s=0$

$= -E(X)$ .... *Equation 7*

Correspondingly,

$L_X^{(k)}(0) = (-1)^k E(X^k)$ .... *Equation 8*

There are many useful properties of Laplace Transform. These properties can make calculations easier when dealing with probability. For instance, let X, Y, Z be three random variables with
Z = X +Y and X, Y are independent.

Then the Laplace Transform of Z can be found as:
$L_Z(s) = L_X(s) \cdot L_Y(s)$ .... *Equation 9*
Moreover, when Z with probability P equals X, with probability 1 − P equals Y, then
$L_Z(s) = PL_X(s) + (1 - P)L_Y(s)$ .... *Equation 10*

Laplace Transforms of some useful distributions can now be introduced.

a. Suppose X is a random variable which follows an exponential distribution with rate λ. The Laplace Transform of X is

$L_X(s) = \frac{(\lambda)}{\lambda + s}$ .... *Equation 11*

b. Suppose X is a random variable which follows an Erlang − r distribution with rate λ. Then X can be written as:

$X = X_1 + X_2 + \cdots + X_r$ .... *Equation 12*

where X_i are i.i.d. exponential with rate λ. Therefore, we have

$LX(s) = LX_1(s) \cdot LX_2(s) \ldots LX_r(s)$

$= \left(\frac{\lambda}{\lambda + s}\right)^n$ .... *Equation 13*

c. Suppose X is a constant real number c, then

$L_X(s) = E(e^{-sX})$

$= E(e^{-sc})$

$= e^{-sc}$ .... *Equation 14*

Basic Queuing Systems

Kendall's notation shall be used to describe a queuing system as denoted by:
A/B/m/K/n/D .... *Equation 15* (Adan and Resing, 2016)
Where

- A: distribution of the interarrival times
- B: distribution of the service times
- m: number of servers
- K: capacity of the system, the maximum number of passengers in the system including the one being serviced
- n: population size of sources of passengers
- D: service discipline

G shall be used to denote general distribution, M used for exponential distribution (M stands for Memory less), D be used for deterministic times (Sztrik, 2016).
A/B/m is also used to describe a queueing system, where:

- A stands for distribution of interarrival times,
- B stands for distribution of service times and
- m stands for number of servers.

Hence M/M/1 denotes a system with Poisson arrivals, exponentially distributed service times and a single server.

M/G/m denotes an m- server system with Poisson arrivals and generally distributed service times, and so on.

In this section, the basic queuing models (M/M/1 system), which is a system with Poisson arrivals, exponentially distributed service times and a single server. The following part is retrieved from Queuing Systems (Adan and Resing, 2016).

Firstly, it is assumed that inter-arrivals follows an exponential distribution with rate λ, and service time follows the exponential distribution with rate μ. Further, in the single service model, to avoid queue length instability, it is assume that:
According to Adanikin, Olutaiwo and Obafemi (2017),

Utilization (R) = $\frac{\text{Average Arrival Rate }(\lambda)}{\text{Average service rate }(\mu)} < 1$ .... *Equation 16*

Here R is the fraction of time the server is working (called the utility factor). Time-dependent behaviour of this system will be considered firstly, then the limiting behaviour. Let $R_n(t)$ denote the probability that at time t there are n passengers in the system.

Then by *equation 3*, when $\Delta t \to 0$,

$R_0(t + \Delta t) = (1 - \lambda\Delta t)R_0(t) + \mu\Delta t R_1(t) + {}_0(\Delta t)$ .... *Equation 17*

$R_n(t + \Delta t) = \lambda\Delta t R_{n-1}(t) + (1 - (\lambda + \mu)\Delta t)R_n(t) + \mu\Delta t R_{n+1}(t) + {}_0(\Delta t)$ .... *Equation 18*

where n = 1,2, ...

Hence, by tending $\Delta t \to 0$, the following infinite set of differential equations for $R_n(t)$ will be obtained.

$R^1{}_0(t) = -\lambda R_0(t) + \mu R_1(t)$ .... *Equation 19*

$R^1{}_n(t) = \lambda p_{n-1}(t) - (\lambda + \mu)R_n(t) + \mu R_{n+1}(t)$, n = 1,2, ... *Equation 20*

It is very difficult to solve these differential equations. However, when we focus on the limiting or equilibrium behaviour of this system, it is much easier.

It was revealed by (Sztrik, 2016) that when $t \to \infty$, $R^1{}_n(t) \to 0$ and $R_n(t) \to R_n$. It follows that the limiting probabilities R_n satisfy equations

$0 = -\lambda R_0 + \mu R_1$............................... *Equation 21*

$0 = \lambda R_{n-1} - (\lambda + \mu)R_n + \mu R_{n+1}$, , n = 1,2, ... *Equation 22*

Moreover, R_n also satisfy

$\sum_{n=0}^{\infty} Rn = 1$ .... *Equation 23*

which is called the normalization equation. We can also use a flow diagram to derive the normalization equations directly. For an M/M/1 system, the flow diagram is shown in figure 4:

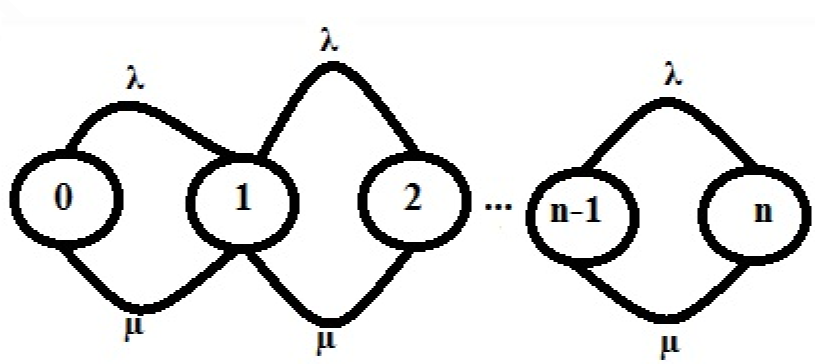

Figure 4 Process diagram for M/M/1 Queue, k=1,2,3,... (Ademoh and Anosike, 2014)

The rate matrix of the system is:

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & . & . & . \\ \mu & -(\mu + \lambda) & \lambda & 0 & 0 & . & . & . \\ 0 & \mu & -(\mu + \lambda) & \lambda & 0 & . & . & . \\ 0 & 0 & \mu & -(\mu + \lambda) & \lambda & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$ .... *Equation 24*

Notice that the sum of each row equals 0.

In order to determine the equations from the flow diagram, a global balance principle was adopted. Global balance principle states that for each set of states under the equilibrium condition, the flow out of set is equal to the flow into that set. Based on figure 1,

$$\begin{pmatrix} State & Rate\ In & = & Rate\ Out \\ 0 & \mu R1 & = & \lambda R0 \\ 1 & \lambda R0 + \mu R2 & = & (\lambda + \mu)R1 \\ 2 & \lambda R1 + \mu R3 & = & (\lambda + \mu)R2 \end{pmatrix}$$

This is exactly the normalization equation. To solve the equation, firstly, it was assume that

(R) = $\frac{Average\ Arrival\ Rate\ (\lambda)}{Average\ service\ rate\ (\mu)}$ which is known as the utilization factor. From the equilibrium equation of state 0, we have:

$R_1 = Rp_0$ .... *Equation 25*

When equation 25 was substituted into the equilibrium equation of state 1, then:

$\lambda p_0 + \mu p_2 = (\lambda + \mu)Rp_0$

$= \frac{\lambda^2}{\mu} p_0$ .... *Equation 26*

That is

$\mu p_2 = \frac{\lambda^2}{\mu} p_0$ .... *Equation 27*

Therefore,

$p_2 = R^2 p_0$ .... *Equation 28*

Generally,

$pk = R^k p_0$ .... *Equation 29*

Since

$\sum_{n=0}^{\infty} p_n = 1$ .... *Equation 30*

Using (1.29), we can replace p_k by R_0. Then

$\sum_{n=0}^{\infty} R^n p_0 = 1$................................. *Equation 31*

That is

$\frac{1}{1-R} p_0 = 1$

$p_0 = 1 - R$.. *Equation 32*

Moreover, for any k,

$p_k = R^k(1 - R)$ .... *Equation 33*

Finally is the limiting probability pk in the M/M/1 system. The expected queue length L is given by

$$E(L) = \sum_{i=0}^{\infty} i.pi$$

$$= \sum_{i=0}^{\infty} i.R^{i}\ (1-R)$$

$$= R(1-R)\sum_{i=0}^{\infty} i.R^{i}$$

$$= R(1-R)\left(\sum_{i=1}^{\infty} [\![(i.R^{i})^{i}]\!]\right.$$

= R(1-R) $\left(\frac{1}{1-R}\right)^{i}$

= $\frac{R}{1-R}$.............................*Equation 34*

3. CONCLUSION

The essence of queuing modeling is to ensure maximum utilization of a system at a minimized constraint. The main objective of a waiting system is to ensure that service rate is greater than the arrival rate such that the system will be regarded as efficient. This study has quantitatively proven the model of a Single-server queue system which occur if arrival and service rate is Poisson distributed (single queue) (M/M/1) queue.

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