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On the ternary biquadratic non-homogeneous equation, (2k + 1)(x2 + y2 + xy) = z4

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ABSTRACT

The ternary biquadratic non-homogeneous equation represented by the diophantine equation

$$(2k+1)(x^2+y^2+xy)=z^4$$

is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

Keywords: Integral solutions,ternary biquadratic non-homogeneous equation ,lattice points.

M.Sc 2000 mathematics subject classification: 11D25

NOTATIONS:

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

S_n : Star number of rank n

$\Pr_n$: Pronic number of rank n

So_n : Stella octangular number of rank n

j_n : Jacobsthal number of rank n

J_n : Jacobsthal lucas number of rank n

Ky_n : Kynea number of rank n

1. INTRODUCTION

The biquadratic diophantine (homogeneous or non-homogeneous) equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-11] for ternary non-homogeneous biquadratic equations. This communication concerns with yet another interesting ternary non-homogeneous biquadratic equation given by

$$(2k+1)(x^2+y^2+xy)=z^4$$

for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS

The diophantine equation representing a non-homogeneous biquadratic equation is

$$(2k+1)(x^2+y^2+xy)=z^4 \tag{1}$$

Introducing the linear transformations,

$$x=(2k+1)(u+v),\ y=(2k+1)(u-v),\ z=(2k+1)w \tag{2}$$

in (1),it leads to

$$3u^2+v^2=(2k+1)w^4 \tag{3}$$

Assume $w = w(a,b) = b^2 + 3a^2$, $a,b \neq 0$ (4)

To find the solution for (3), choose k such that

$$(2k+1) = B^2 + 3A^2 \text{ , where } B = 2p+1, A = 2q \quad (5)$$

Employing the method of factorization, define

$$(v + i\sqrt{3}u) = \left((2p+1) + i(\sqrt{3}2q)\right)(b + i\sqrt{3}a)^4 \quad (6)$$

Equating real and imaginary parts in (6) we get,

$$v(a,b,p,q) = (2p+1)(b^4 + 9a^4 - 18a^2b^2) - 6q(4ab^3 - 12ab) \quad (7)$$

$$u(a,b,p,q) = (2p+1)(4ab^3 - 12ab) + 2q(b^4 + 9a^4 - 18a^2b^2) \quad (8)$$

Substituting (7) and (8) in (2), the corresponding integral solutions of (1) are presented by

$$x = x(a,b,p,q) = (2k+1)\begin{pmatrix} (2p+1)(4ab^3 - 12ab + b^4 + 9a^4 - 18a^2b^2) \\ + 2q(b^4 + 9a^4 - 18a^2b^2 - 12ab^3 + 36ab) \end{pmatrix}$$

$$y = y(a,b,p,q) = (2k+1)\begin{pmatrix} (2p+1)(4ab^3 - 12ab - b^4 - 9a^4 + 18a^2b^2) \\ + 2q(b^4 + 9a^4 - 18a^2b^2 + 12ab^3 - 36ab) \end{pmatrix}$$

$$z = z(a,b,p,q) = \left((2p+1)^2 + 3(2q)^2\right)(b^2 + 3a^2)$$

2.1. Properties

1. $x(1,b,p,q) + y(1,b,p,q) - 4(2k+1)(2p+1)So_b - (2k+1)(2q)(2t_{4,b}^2 - 36t_{4,b} + 18) \equiv 0(\bmod 20)$
2. $x(a,1,p,q) - y(a,1,p,q) - (2k+1)(2p+1)(2S_a + 24t_{3,a} - 60t_{4,a} + 18t_{4,a}^2) \equiv 0(\bmod 48)$
3. Each of the following is a nasty number.

(i) $6z(a,a,6k^2 + 6k - 2l^2 + 1,2l(2k+1)$

(ii) $6z(a,a,6l^2 - 2k^2 - 2k - 1,2l(2k+1)$, When p and q are different pairty.

4. $y(1,b,p,q) - (2k+1)(2p+1)(36P_b^3 - 4t_{3,b}^2 + (b-12)^2 - 153) -$

$$(2k+1)(2q)(4_{3,b}^2 + P_b^6 + 3So_b - 32\,\mathrm{Pr}_b - 10t_{4,b} + 9) \equiv 0$$

It is observed that, in (5), one may also take $B = 2p, A = 2q+1$ .For this choice, the corresponding integral solutions of (1) are obtained as

$$x = x(a,b,p,q) = (2k+1)\left((2p+1+2q)(b^4 + 9a^4 - 18a^2b^2) + (2p - 6q - 3)(4ab^3 - 12ab)\right)$$

$$y = y(a,b,p,q) = (2k+1)\left((2q+1-2p)(b^4 + 9a^4 - 18a^2b^2) + (2p + 6q + 3)(4ab^3 - 12ab)\right)$$

$$z = z(a,b,p,q) = \left((2p)^2 + 3(2q+1)^2\right)(b^2 + 3a^2)$$

3. REMARKABLE OBSERVATIONS

I.: Let (x_0, y_0, z_0) be any given non- zero solution of (1).Then each of the following triples

$(x_{2n-1}, y_{2n-1}, z_{2n-1}) = (a^{4n-2}y_0, a^{4n-2}x_0, a^{2n-1}z_0)$, $(x_{2n}, y_{2n}, z_{2n}) = (a^{4n}x_0, a^{4n}y_0, a^{2n}z_0)$ also satisfy (1).

A few interesting relations observed from the above triples are presented below.

1. $$\frac{x_{2n-1}}{y_0} = \frac{y_{2n-1}}{x_0} = \left(\frac{z_{2n-1}}{z_0}\right)^2$$

2. $\left(\frac{x_{2n-1}}{y_0}, \left(\frac{z_{2n-1}}{z_0}\right)^2, \frac{y_{2n-1}}{x_0}\right)$ forms an Arithmetic Progression

3. $\left(\frac{x_{2n}}{x_0}, \left(\frac{z_{2n}^2}{z_0^2}\right), \frac{y_{2n}}{y_0}\right)$ forms an Arithmetic Progression

4. Each of the following is a nasty number.

(a) $6\left(\frac{x_{2n}}{x_0}\frac{y_{2n}}{y_0}\right)$

(b) $6\left(\frac{x_{2n-1}}{y_0}\frac{y_{2n}}{y_0}\right)$

(c) $6\left(\frac{x_{2n}}{x_0}\frac{y_0}{x_{2n-1}}\right)$

(d) $6\left(\frac{x_{2n-1}}{y_0}\frac{y_{2n-1}}{x_0}\right)$

5. Each of the following is a cubical integer.

(a) $a^2\left(\frac{y_{2n-1}}{x_0}\frac{z_{2n}}{z_0}\right)$

(b) $a^2\left(\frac{x_{2n-1}}{y_0}\frac{z_{2n}}{z_0}\right)$

6. Each of the following is a biquadratical integer.

(a) $a\left(\frac{z_{2n-1}}{z_0}\frac{z_{2n}}{z_0}\right)$

(b) $a^2\left(\frac{x_{2n-1}}{y_0}\frac{y_{2n}}{y_0}\right)$

7. In particular, when $a = 2^k$, $k > 1$,a few results observed are as follows:

(a) $\frac{y_{2n}}{y_0} = 3J_{4kn} + 1$

(b) $\frac{z_{2n}}{z_0} = j_{2kn} - 1$

(c) $\frac{z_{4n}}{z_0} + 2\frac{z_{2n}}{z_0} - 1 = ky_{2n}$

(d) $\left(\frac{2^{4k}-1}{2^{4k}}\right)\left(\sum_{n=1}^{N}\frac{x_{2n}}{x_0}\right) + 1 = (j_{2k} - 1)\frac{x_{2n-1}}{y_0} = (3J_{2k} + 1)\frac{y_{2n-1}}{x_0} = \frac{x_{2n}}{x_0}\frac{x_{2n-1}}{y_0}$

4. CONCLUSION

To conclude, one may search for other pattern of solutions and their corresponding properties.

REFERENCES

1. Dickson. I. E. “History of the Theory of Numbers’’, Vol 2. Diophantine analysis, New York, Dover,2005.
2. Mordell .L J., “ Diophantine Equations Academic Press’’, New York,1969.
3. Carmichael. R.D. “ The Theory of numbers and Diophantine Analysis’’ ,New York, Dover, 1959.
4. Gopalan.M.A., Manjusomnath and Vanith N., “Parametric integral solutions of $x^2 + y^3 = z^4$”, Acta Ciencia Indica,Vol.XXXIIIM(No.4)Pg.1261-1265(2007).
5. Gopalan.M.A.,and Pandichelvi .V., “On Ternary biquadratic Diophantine equation $x^2 + ky^3 = z^4$”, Pacific- Asian Journal of Mathematics, Vol-2.No.1-2, 57-62, 2008.

6. Gopalan.M.A., and Janaki.G., "Observation on $2(x^2 - y^2) + 4xy = z^4$", Acta Ciencia Indica,Vol.XXXVM(No.2)Pg.445-448,(2009).

7. Gopalan.M.A., Manjusomnath and Vanith N., "Integral solutions of $x^2 + xy + y^2 = (k^2 + 3)^n z^4$", Pure and Applied Mathematika Sciences,Vol.LXIX,NO.(1-2):149-152,(2009).

8. Gopalan.M.A., and Sangeetha.G., "Integral solutions of ternary biquadratic equation $(x^2 - y^2) + 2xy = z^4$", Antartica J.Math.,7(1)Pg.95-101,(2010).

9. Gopalan.M.A.,and Janaki.G., "Observations on $3(x^2 - y^2) + 9xy = z^4$", Antartica J.Math.,7(2)Pg.239-245,(2010).

10. Gopalan.M.A., and Vijayasankar.A., "Integral Solutions of Ternary biquadratic equation $x^2 + 3y^2 = z^4$", Impact.J.Sci.Tech.Vol.4(No.3)Pg.47-51,2010.

11. Gopalan.M.A,Vidhyalakshmi.S., and Devibala.S., "Ternary biquadratic diophantine equation $2^{4n+3}(x^3 - y^3) = z^4$",Impact. J.Sci.Tech.Vol4(No.3)Pg.57-60,2010.