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# Calculated the ground level concentration by using the Laplace technique and compare them with other ways

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## ABSTRACT

In this paper, the ground level concentration of air pollutants is obtained by using Laplace technique to solve the diffusion equation in two dimensions. Considering that both the wind speed and eddy diffusivity in z-direction are constant. Comparing between outcomes values with data measured at Copenhagen, Denmark, Gaussian model and also with separation of variables technique which have been calculated in previous paper.

**Keywords:** Diffusion Equation, Separation of Variables, Gaussian Model, Laplace Technique

**Abbreviations**: ANOVA - one way analysis of variance.

## 1. INTRODUCTION

The analytical solution of the atmospheric diffusion equation contains different depending on Gaussian and non-Gaussian solutions. An analytical solution with power law of the wind speed and eddy diffusivity with the realistic assumption is solved (Essa and Fouad, 2011; and Demuth 1978). In an attempt to study the essential physics of shear-flow dispersion, several investigators have derived solutions to vary with height (e.g. Report, personal communication in (Sutton, 1953; Smith, 1957; and Marrouf, et al., 2013). The study of the diffusion of pollutants in the atmosphere from different sources depend on the concepts of physics describes the movement of pollutants in the atmosphere and the effect of the boundary layer. The heart of any atmospheric diffusion model is to estimate the concentration of pollutants by calculating from some basic information about the source of the pollutants and the metrological condition (Karl et al., 2000; and Alharbi et al., 2011; Dagar and Shakuntla Devi Dagar, 2015). Gaussian plume models are commonly used for the air quality analysis and regulatory purposes, and based on the solution of the diffusion equation such that the wind speed and eddy diffusivity are constant (khaled et al., 2011). The Gaussian plume model are widely used, well understood, easy to apply, and until more recently have received international approval (Ross et al., 2001; Abdel-wahab et al., 2014). Laplace transformation technique has been used to get desired solutions, in addition to this method, Hankle transform method, Airs moment method, perturbation approach, and other methods have also been used to get the analytical solutions of the advection – diffusion equation in one, two and three dimensions. But Laplace transformation technique has been commonly used because of being simpler than other methods and the analytical solutions using this method being more reliable in verifying the numerical solutions in terms of the accuracy and stability (Abdel-Wahab et al., 2014). The purpose of this study is to suggest a simple physical realistic model depend on the atmosphere over which diffusion of pollution takes place, also, wind speed is treated as a function of height and stability of the atmosphere. This definitely more closely represents real life situations than does treating wind as a constant quantity (khaled SM Essa et al., 2013).

## 2. MATHEMATICAL MODEL

The important main in the dispersion of pollutants in the atmosphere is the atmospheric diffusion equation which based on the gradient transport theory .then the diffusion equation of pollutants in air can be written as (Arya, 1995).

$$u\,\frac{\partial c(x,y,z)}{\partial x} = \frac{\partial}{\partial y}\left(k_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right) \quad (1)$$

Where: u is the wind speed (m/s).

For simplicity $k_z = k(x)$ is the eddy diffusivity

By integration with respect to y from $-\infty$ to $\infty$, then we get

$$u\,\frac{\partial}{\partial x}\int_{-\infty}^{\infty} c(x,y,z)dy = k_y\left.\frac{\partial c(x,y,z)}{\partial y}\right| + k(x)\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty} c(x,y,z)dy$$
(2)

Suppose that.

$$\int_{-\infty}^{\infty} c(x,y,z)dy = c_y(x,z)$$
(3)

One has that

$$k_y\left.\frac{\partial c(x,y,z)}{\partial y}\right| = 0$$
(4)

*By substituting from Eq* (3)&(4)*into* (2)*one can get*: –

$$u\frac{\partial c_y(x,z)}{\partial x} = \frac{\partial}{\partial z}\left[k(x)\frac{\partial c_y(x,z)}{\partial z}\right]$$
(5)

$$u\frac{\partial c_{y(x,z)}}{\partial x} = k(x)\frac{\partial^2 c_y(x,z)}{\partial z^2}$$
(6)

By using the equation (6), taking Laplace transform on x then.

$$\tilde{c}(s,z)=\int_0^\infty c(x,z)e^{-sx}dx$$
(7)

$$\int_0^\infty u\frac{\partial c}{\partial x}e^{-sx}dx = k\int_0^\infty \frac{\partial^2 c}{\partial z^2}e^{-sx}dx$$
(8)

Integrating and substituting in the equation (8), one can get:-

$$c(0,z)+s\tilde{c}(s,z) = \frac{k}{u}\frac{\partial^2 \widetilde{c(s,z)}}{\partial z^2}$$
(9)

Let $u\, c(0,z)=Q\delta(z-h_s)$
(10)

Substituting in equation (9) one can get:-

$$\frac{\partial^2 \tilde{c}_y(s,z)}{\partial z^2} - \frac{su}{k}\tilde{c}(s,z)=-\frac{Q}{u}\delta(z-h_s)\frac{u}{k}$$
(11)

Then the equation (11) can be written as:-

$$\frac{\partial^2}{\partial z^2}\tilde{c}_y(s,z) - \frac{su}{k}\tilde{c}_y(s,z) = -\frac{Q}{k}\delta(z-h_s)$$
(12)

This equation is non-homogeneous the general solution is the sum of the solution of non-homogeneous and homogenous equation then, the solution of homogenous equation can be written as:-

$$\tilde{c}_y(s,z)=c_1e^{z\sqrt{\frac{su}{k}}} + c_2e^{-z\sqrt{\frac{su}{k}}}$$
(13)

Using the condition the concentration is bounded at infinity i.e.

$c_y(s,z)=0$ at $z=\infty$ then we find $C_1=0$.

$\therefore\ \tilde{c}_y(s,z) = c_2 e^{-z\sqrt{\frac{su}{k}}}$ (14)

Taking Laplace transform on equation (10) one gets:-

$$\tilde{c}_y(s,z) = \frac{Q}{us}\delta(z - h_s) \quad (15)$$

where

$$L\left[\frac{\partial c_y(x,z)}{\partial x}\right] = s\{\tilde{c}_y(s,z)\} - c_y(0,z)$$

Substituting in equation (14)

$$c_2 = \frac{Q}{us}\delta(z - h_s) \quad (16)$$

Substituting from equation (16) in equation (14) then the solution of the homogenous equation can be written as:

$$\tilde{c}_y(s,z) = \frac{Q}{us}\delta(z - h_s)e^{-z\sqrt{\frac{su}{k}}} \quad (17)$$

The solution of nonhomogeneous equation (12) can be written as:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{su}{k}\right)\tilde{c}_y(s.z) = -\frac{q}{k}\delta(z - h_s) \quad (18)$$

$$\left(D-\sqrt{\frac{su}{k}}\right)\left(D+\sqrt{\frac{su}{k}}\right)\tilde{c}_y(s,z) = -\frac{q}{k}\delta(z - h_s) \quad (19)$$

Let us know the theorem in the form: $\frac{1}{(D-m)}R(z)$ is the special solution then, one divides the equation (19) into two parts.

i- $\left(D-\sqrt{\frac{su}{k}}\right)\tilde{c}_1(s,z) = -\frac{q}{k}\delta(z - h_s)$ then, the equation becomes:-

$$\tilde{c}_1(s,z) = \frac{-q}{k\left(D-\sqrt{\frac{su}{k}}\right)}\delta(z-h_s)$$

(20)

Substituting the theorem as follows:-

$$\frac{1}{(D-m)}R(z) = e^{mz}\int e^{-mz}R(z)dz$$
(21)

Then solution of equation (20) can be written in the form:-

$$\tilde{c}_1(s,z) = -\frac{Q}{k}e^{z\sqrt{\frac{su}{k}}}\int e^{-z\sqrt{\frac{su}{k}}}\delta(z-h_s)dz$$
(22)

$$\tilde{c}_1(s,z) = -\frac{Q}{K}e^{z\sqrt{\frac{su}{k}}}\left(e^{-h_s\sqrt{\frac{su}{k}}}\right) = -\frac{Q}{K}e^{(Z-h_s)\sqrt{\frac{su}{k}}}$$

(23)

ii- $(D+\sqrt{\frac{su}{k}})\ \tilde{c}_y(s,z) = \tilde{c}_1(s,z)$

(24)

Then applying the theorem, one gets:-

$$\tilde{c}_y(s,z) = -\frac{Q}{k}e^{-z\sqrt{\frac{su}{k}}}\int_0^h e^{z\sqrt{\frac{su}{k}}}\, e^{(z-h_s)\sqrt{\frac{su}{k}}}\,dz$$
(25)

where "h" is mixing height and "$h_s$" is stack height.

Integrating with respect to "z" and substituting then the equation becomes.

$$\tilde{c}_y(s,z) = -\frac{Q}{2\sqrt{SUK}}\left[e^{-\sqrt{\frac{su}{k}}(z-2h+h_s)} - e^{-\sqrt{\frac{su}{k}}(z+h_s)}\right]$$

(26)

The solution of equation (12) is the sum of equation (17) and equation (27) as follows:-

$$\frac{\tilde{c}_y(s,z)}{Q} = \frac{1}{us}\delta(z-h_s)e^{-z\sqrt{\frac{su}{k}}} + \frac{1}{2\sqrt{suk}}\left(e^{-\sqrt{\frac{su}{k}}(z+h_s)}\right) - \frac{1}{2\sqrt{suk}}\left(e^{-\sqrt{\frac{su}{k}}(z-2h+h_s)}\right) \quad (27)$$

Taking the inverse Laplace transform for the equation (28), one can get.

$$\frac{c_y(x,z)}{Q} = e^{-\frac{u(z+h_s)^2}{4kx}}\left[\frac{1}{u\pi x}+\frac{1}{2\sqrt{u\pi kx}}\right] - \frac{1}{2\sqrt{u\pi kx}}e^{\frac{-u(z+2h-h_s)^2}{4kx}}$$

Calculating the concentration at ground level surface under centerline of the plume put z=0

$$\frac{c_y(x,0)}{Q} = e^{-\frac{uh_s{}^2}{4kx}}\left[\frac{1}{u\pi x}+\frac{1}{2\sqrt{u\pi kx}}\right] - \frac{1}{2\sqrt{u\pi kx}}e^{-\frac{u(2h-h_s)^2}{4kx}} \quad (28)$$

Taking the eddy diffusivity as follows:- k(x)=$\alpha ux$, where u is the mean wind speed ,α is the parameter for turbulence intensity, such that $\alpha=\left(\frac{\sigma_w}{u}\right)^2$, $\sigma_w$ is the standard deviation of the vertical velocity

$$k(x) = \frac{\sigma_w{}^2}{u}x$$

## 3. VALIDATION

The used data was observed from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark, under neutral and unstable conditions (Gryning et al., 1984; and Gryning et al., 1987).

**Table 1**

Comparison between observed, predicted with separation of variable model, Gaussian model and Laplace technique integrated crosswind ground level concentrations under unstable condition with downwind distance

| Run no | Date | PG Stability | H (m) | $U_{10}$ $(ms^{1})$ | $\sigma_w$ $(ms^{1})$ | Distance (m) | $C_y/Q$ $(10^{-4}\ sm^{-2})$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | Observed | | | Computed |
| | | | | | | | | separation | Gaussian | Laplace |
| 1 | 20-9-78 | A | 1980 | 2.1 | 0.83 | 1900 | 6.48 | 7.17 | 5.16 | 7,704693 |
| 1 | 20-9-78 | A | 1980 | 2.1 | 0.83 | 3700 | 2.31 | 5.13 | 2.52 | 3,488227 |
| 2 | 26-9-78 | C | 1920 | 4.9 | 1.07 | 2100 | 5.38 | 3.7 | 2.29 | 4,61996 |
| 2 | 26-9-78 | C | 1920 | 4.9 | 1.07 | 4200 | 2.95 | 2.18 | 1.18 | 2,306918 |
| 3 | 19-10-78 | B | 1120 | 2.4 | 0.68 | 1900 | 8.20 | 9.8 | 4.51 | 8,410968 |
| 3 | 19-10-78 | B | 1120 | 2.4 | 0.68 | 3700 | 6.22 | 7.53 | 2.65 | 3,220596 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 19-10-78 | B | 1120 | 2.4 | 0.68 | 5400 | 4.30 | 7.44 | 2.58 | 1,613861 |
| 4 | 3-11-78 | C | 1390 | 2.5 | 0.47 | 4000 | 11.7 | 7.11 | 6.29 | 5,228509 |
| 5 | 9-11-78 | C | 820 | 3.1 | 0.71 | 2100 | 6.72 | 9.30 | 3.63 | 6,580095 |
| 5 | 9-11-78 | C | 820 | 3.1 | 0.71 | 4200 | 5.84 | 7.87 | 2.44 | 2,044103 |
| 5 | 9-11-78 | C | 820 | 3.1 | 0.71 | 6100 | 4.97 | 7.86 | 2.41 | 1,00388 |
| 6 | 30-4-78 | C | 1300 | 7.2 | 1.33 | 2000 | 3.96 | 3.57 | 1.63 | 3,751729 |
| 6 | 30-4-78 | C | 1300 | 7.2 | 1.33 | 4200 | 2.22 | 2.50 | 0.82 | 1,703804 |
| 6 | 30-4-78 | C | 1300 | 7.2 | 1.33 | 5900 | 1.83 | 2.20 | 0.68 | 1,005404 |
| 7 | 27-6-78 | B | 1850 | 4.1 | 0.87 | 2000 | 6.70 | 5.27 | 2.51 | 5,917179 |
| 7 | 27-6-78 | B | 1850 | 4.1 | 0.87 | 4100 | 3.25 | 3.52 | 1.17 | 2,893086 |
| 7 | 27-6-78 | B | 1850 | 4.1 | 0.87 | 5300 | 2.23 | 3.06 | 0.97 | 2,123662 |
| 8 | 6-7-78 | D | 810 | 4.2 | 0.72 | 1900 | 4.16 | 8.39 | 4.20 | 7,124995 |
| 8 | 6-7-78 | D | 810 | 4.2 | 0.72 | 3600 | 2.02 | 6.21 | 2.80 | 3,123895 |
| 8 | 6-7-78 | D | 810 | 4.2 | 0.72 | 5300 | 1.52 | 5.89 | 2.18 | 1,518876 |
| 9 | 19-7-78 | C | 2090 | 5.1 | 0.98 | 2100 | 4.58 | 3.43 | 2.20 | 4,902058 |
| 9 | 19-7-78 | C | 2090 | 5.1 | 0.98 | 4200 | 3.11 | 2.77 | 1.13 | 2,485021 |
| 9 | 19-7-78 | C | 2090 | 5.1 | 0.98 | 6000 | 2.59 | 2.49 | 0.81 | 1,682239 |

Table (1) shows that the comparison between observed, predicted with separation of variable model, Gaussian model and Laplace technique integrated crosswind ground level concentrations under unstable condition with downwind distance.

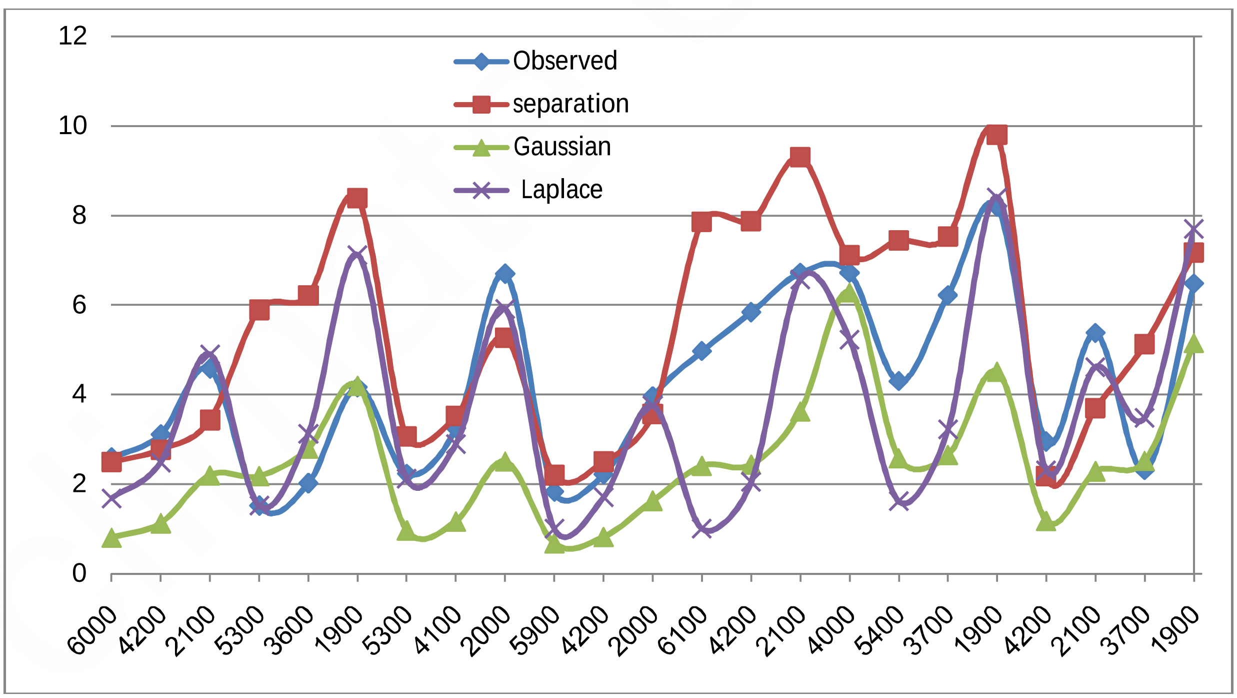


**Figure 1**
The variation of the three predicted and observed models via downwind distances.

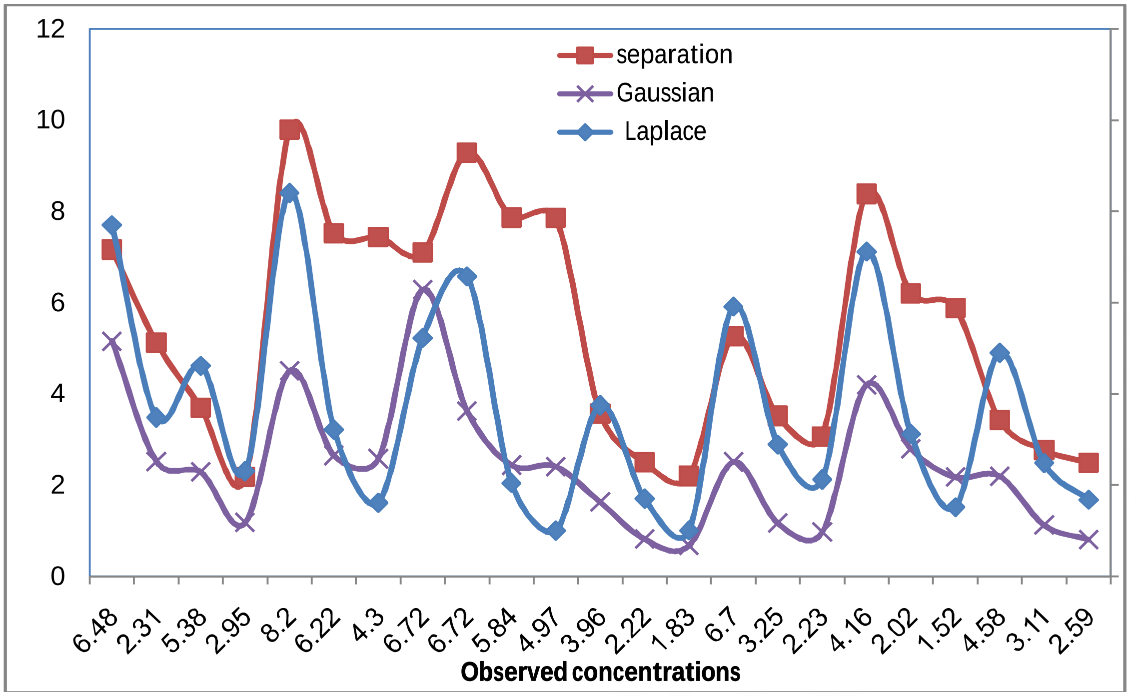


**Figure 2**
The variation between the predicted models and observed concentrations data

## 4. MODEL EVALUATION STATISTICS

Now, the statistical method is presented and comparison between predicted and observed results will be offered by (Hanna, 1989). The following standard statistical performance measures that characterize the agreement between prediction ($C_p = C_{pred}/Q$) and observations ($C_o = C_{obs}/Q$):

$$\text{Fractional Bias (FB)} = \frac{\left(\overline{C_o} - \overline{C_p}\right)}{\left[0.5\left(\overline{C_o} + \overline{C_p}\right)\right]}$$

$$\text{Normalized Mean Square Error (NMSE)} = \frac{\overline{\left(C_p - C_o\right)^2}}{\left(\overline{C_p}\,\overline{C_o}\right)}$$

$$\text{Correlation Coefficient (COR)} = \frac{1}{N_m}\sum_{i=1}^{N_m}\left(C_{pi} - \overline{C_p}\right) \times \frac{\left(C_{oi} - \overline{C_o}\right)}{\left(\sigma_p \sigma_o\right)}$$

$$\text{Factor of Two (FAC2)} = 0.5 \le \frac{C_p}{C_o} \le 2.0$$

where $\sigma_p$ and $\sigma_o$ are the standard deviations of $C_p$ and $C_o$ respectively. Here the over bars indicate the average over all measurements. A perfect model would have the following idealized performance: NMSE = FB = 0 and COR= FAC2 = 1.0.

**Table 2**
Comparison between Laplace, Separation and Gaussian models according to standard statistical Performance measure

| Models | NMSE | FB | COR | FAC2 |
|---|---|---|---|---|
| Laplace model | 0.18 | 0.10 | 0.64 | 0.95 |
| Separation model | 0.22 | -0.19 | 0.60 | 1.38 |
| Gaussian model | 0.58 | 0.58 | 0.80 | 0.59 |

## 5. DISCUSSION

Figure (1) Shows that the predicted normalized crosswind integrated concentrations values of the Laplace and separation predicted models are well to the observed data than the Gaussian predicted model. Figure (2) Shows that the Laplace and separation predicted methods data are nearer to the observed concentrations data than the predicted Gaussian model. From the above figures, we find that there is agreement between the Laplace and separation predicted normalized crosswind integrated concentrations with the observed normalized crosswind integrated concentrations than the predicted Gaussian model. From the statistical method, we find that the three models are inside a factor of two with observed data. Regarding to NMSE and FB, the Laplace and separation predicted models are well with observed data than the Gaussian model. The correlation of Laplace and separation predicated model equals (0.64 and 0.60 respectively) and Gaussian model equals (0.80).

## 6. CONCLUSION

The ground level concentration of air pollutants is obtained by using Laplace technique to solve the diffusion equation in three dimensions. Considering that both the wind speed and eddy diffusivity in z-direction are constant. One finds that there is agreement between the Laplace and separation predicted normalized crosswind integrated concentrations with the observed normalized crosswind integrated concentrations than the predicted Gaussian model. From the statistical method, one finds that the three models are inside a factor of two with observed data. Regarding to NMSE and FB, the Laplace and separation predicted models are well with observed data than the Gaussian model. The correlation of Laplace and separation predicated model equals (0.64 and 0.60 respectively) and Gaussian model equals (0.80).

## DISCLOSURE STATEMENT

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